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Analytic Solutions of (N+1) Dimensional Time Fractional Diffusion Equations by Iterative Fractional Laplace Transform Method

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ABSTRACT

In this paper, the author has been examined how to obtain solutions of (n+1)dimensional time fractional diffusion equations with initial conditions in the form of infinite fractional power series, in terms of Mittage Lefler function of one parameter and exact form by the use of iterative fractional Laplace transform method(IFLTM). The basic idea of the IFLTM was developed successfully. To see its effectiveness and applicability, three test examples were presented. The closed solutions in the form of infinite fractional power series and in terms of Mittag-Leffler functions in one parameter, which rapidly converge to exact solutions, were successfully derived analytically by the use of iterative fractional Laplace transform method (IFLTM).Thus, the results show that the iterative fractional Laplace transform method works successfully in solving (n+1)dimensional time fractional diffusion equations in a direct way without using linearization, perturbation, discretization or restrictive assumptions, and hence it can be extended to other fractional differential equations.

Keywords: *(n+1)Dimensional Time Fractional Diffusion equations with initial conditions, Caputo fractional Derivatives, Mittag-Leffler function, Iterative Fractional Laplace Transform Method.*

INTRODUCTION

Fractional calculus theory is a mathematical analysis tool to the study of integrals and derivatives of arbitrary order, which unify and generalize the notations of integer-order differentiation and $n-$fold integration (El-Ajou, Arqub, Al-Zhour, & Momani, 2013; Millar & Ross, 1993; Oldham & Spanier, 1974; Podlubny, 1999).

The L'Hopital's letter raised the question "What does $\frac{\partial mf(x)}{\partial x^m}$ mean if $m=\frac{1}{2}$?" to Leibniz in 1695 is considered to be where the idea of fractional calculusstarted(Diethelm, 2010; Hilfer, 2000; Lazarevic, et al., 2014; Millar & Ross, 1993; Kumar & Saxena, 2016). Since then, much works on this question and other related questions have done up to the middle of the 19th century by many famous mathematicians such as Laplace, Fourier, Abel, Liouville, Riemann, Grunwald, Letnkov, Levy, Marchaud, Erdelyi andReiszand these works sum up leads to contribution screating the field which is known today as fractional calculus(Oldham & Spanier, 1974).

Even though fractional calculus is nearly as old as the standard calculus, it was only in recent few decades that its theory and applications have rapidly developed. It was Ross who organized the first international conference on fractional calculus and its applications at the University of new Haven in June 1974, and edited the proceedings(Ross, 1975). Oldham and Spanier (1974) published the first monograph on fractional calculus in 1974. Next, because of the fact that fractional derivatives and integrals are non-local operators and then this property make them a powerful instrument for the description of memory and hereditary properties of different substances(Podlubny, 1999),theory and applications of fractional calculus have attracted much interest and become a pulsating research area.

Due to this, fractional calculus has got important applications in different fieldsof science, engineering and finance. For instance, Shanantu Das(2011) discussed that fractional calculusis applicable to problems in:fractance circuits, electrochemistry, capacitor theory, feedback control system, vibration damping system, diffusion process, electrical science, and material creep. Podlubny(1999)discussed that fractional calculus isapplicable to problems in fitting experimental data, electric circuits, electro-analytical chemistry, fractional multi-poles, neurons and biology(Podlubny, 1999).Fractional calculus is also applicable to problems in: polymer science, polymer physics, biophysics, rheology, and thermodynamics(Hilfer, 2000).In addition, it is applicable to problems in: electrochemical process(Millar & Ross, 1993; Oldham & Spanier, 1974; Podlubny, 1999) , control theory(David, Linarese, & Pallone, 2011; Podlubny, 1999), physics(Sabatier, Agrawal, & Machado, 2007), science and engineering(Kumar & Saxena, 2016), transport in semi-infinite medium(Oldham & Spanier, 1974), signal processing(Sheng, Chen, & Qiu, 2011), food science (Rahimy, 2010), food gums (David & Katayama, 2013), fractional dynamics(Tarsov, 2011; zaslavsky, 2005), modeling Cardiac tissue electrode interface(Magin, 2008), food engineering and econophysics (David, Linarese, & Pallone, 2011), complex dynamics in biological tissues(Margin, 2010), viscoelasticity(Dalir & Bashour, 2010; Mainardi, 2010; Podlubny, 1999; Rahimy, 2010; Sabatier, Agrawal, & Machado, 2007), modeling oscillation systems(Gomez-Aguilar, Yepez-Martinez, Calderon-Ramon, Cruz-Orduna, Escobar-Jimenez, & Olivares-Peregrino, 2015). Some of these mentioned applications were tried to be touchedas follows.

In the area of science and engineering, different applications of fractional calculus have been developed in the last two decades. For instance, fractional calculus was used in image processing, mortgage, biosciences, robotics, motion of fractional oscillator and analytical science(Kumar & Saxena, 2016). It was also used to generalize traditional classical inventory model to fractional inventory model (Das & Roy, 2014).

In the area of electrochemical process, for example half-order derivatives and integrals proved to be more useful for the formulation of certain electrochemical problems than the classical models(Millar & Ross, 1993; Oldham & Spanier, 1974; Podlubny, 1999).

In the area of viscoelasticity, the use of fractional calculusfor modeling viscoelastic materials is well known. For viscoelastic materials the stress-strain constitutive relation can be more accurately described by introducing the fractional derivative(Carpinteri, Cornetti, & sapora, 2014; Dalir & Bashour, 2010; Duan, 2016; Koeller, 1984; Mainardi, 2010; Podlubny, 1999).

Fractional derivatives, which are the one part of fractional calculus, are used to name derivatives of an arbitrary order(Podlubny, 1999). Recently, fractional derivatives have been successfully applied to describe (model) real world problems.

In the area of physics, fractional kinetic equations of the diffusion, diffusion-advection and Focker-Plank type are presented as a useful approach for the description of transport dynamics in complex systems that are governed by anomalous diffusion and non-exponential relaxation patterns (Metzler & Klafter, 2000). Metzler and Klafter (2000) derived these fractional equations asymptotically from basic random walk models, and from a generalized master equation. They presented an integral transformation between the Brownian solution and its fractional counterparts. Moreover, a phase space model was presented to explain the genesis of fractional dynamics in trapping systems. These issues make the fractional equation approach powerful. Their work demonstrates that the fractional equations have come of age as a complementary tool in the description of anomalous transport processes.L.R. Da Silva, Tateishi, M.K. Lenzi, Lenzi and Da silva(2009)were also discussed that solutions for a system governed by a non-Markovian Fokker Planck equation and subjected to a Comb structure were investigated by using the Green function approach. This structure consists of the axis of structure as the backbone and fingers which are attached perpendicular to the axis, and for this system, an arbitrary initial condition in the presence of time dependent diffusion coefficients and spatial fractional derivatives was considered and the connection to the anomalous diffusion was analyzed (L.R. Da Silva et al., 2009).

In addition to these, the following are also other applications of fractional derivatives. Fractional derivatives in the sense of Caputo fractional derivatives were used in generalizing some theorems of classical power series to fractional power series (El-Ajou et al., 2013). Fractional derivative in the Caputo sense was used to introduce a general form of the generalized Taylor’s formula by generalizing some theorems related to the classical power series into fractional power series sense (El-Ajou, Abu Arqub, & Al-S, 2015). A definition of Caputo fractional derivative proposed on a finite interval in the fractional Sobolev spaces was investigated from the operator theoretic viewpoint(Gorenflo, Luchko, & Yamamoto, 2015). Particularly, some important equivalence of the norms related to the fractional integration and differentiation operators in the fractional Sobolev spaces are given and then applied for proving the maximal regularity of the solutions to some initial-boundary-value problems for the time-fractional diffusion equation with the Caputo derivative in the fractional Sobolev spaces(Gorenflo, Luchko, & Yamamoto, 2015).With the help of Caputo time-fractional derivative and Riesz space-fractional derivative, the α -fractional diffusion equation, which is a special model for the two-dimensional anomalous diffusion, is deduced from the basic continuous time random walk equations in terms of a time- and space-fractional partial differential equation with the Caputo time-fractional derivative of order $\frac{\alpha}{2}$ and the Riesz space-fractional derivative of order α (Luchko, 2016). Fractional derivatives were also used to describe HIV infection of $CD4^{+}T$ with therapy effect (Zeid, Yousefi, & Kamyad, 2016).

In the area of modeling oscillating systems, caputo and Caputo-Fabrizio fractional derivatives were used to present fractional differential equations which are generalization of the classical mass-spring-damper model, and these fractional differential equations are used to describe variety of systems which had not been addressed by the classical mass-spring-damper model due to the limitations of the classical calculus (Gomez-Aguilaret al., 2015).

Podlubny(1999)stated that fractional differential equations are equations which contain fractional derivatives. These equations can be divided into two categories such as fractional ordinary differential equations and fractional partial differential equations. Fractional partial differential equations (PDES) are a type of differential equations (DEs) that involving multivariable function and their fractional or fractional-integer partial derivatives with respect to those variables(Abu Arqub, El-Ajou, & Momani, 2015).There are different examples of fractional partial differential equations. Some of them are: the time-fractional Boussinesq-type equation, the time-fractional $B(2,1,1)$ -type equation and the time-fractional Klein-Gordon-type equation stated in Abu Arqub et al.(2015), and time fractional diffusion equation stated inA.Kumar, kumar and Yan (2017), Cetinkaya and Kiymaz (2013), Kumar, Yildirim, Khan and Wei(2012) and so on.

Recently, fractional differential equations have been successfully applied to describe (model) real world problems. For instance, the generalized wave equation, which contains fractional derivatives with respect to time in addition to the second-order temporal and spatial derivatives, was used to model the viscoelastic case and the pure elastic case in a single equation(Wang, 2016).The time fractional Boussinesq-type equations can be used todescribe small oscillations of nonlinear beams, long waves over an even slope, shallow-water waves, shallow fluid layers, and nonlinear atomic chains; the time-fractional $B(2,1,1)$ -type equations can be used to study optical solitons in the two cycle regime, density waves in traffic flow of two kinds of vehicles, and surface acoustic soliton in a system supporting love waves; the time fractional Klein-Gordon-type equations can be applied to study complex group velocity and energy transport in absorbing media, short waves in nonlinear dispersive models, propagation of dislocations within crystals(As cited in Abu Arqub et al., 2015).As cited in Abu Arqub(2017), the time-fractional heat equation, which is derived from Fourier’s law and conservation of energy, is used in describing the distribution of heat or variation in temperature in a given region over time; the time-fractional cable equation, which is derived from the cable equation for electro diffusion in smooth homogeneous cylinders and occurred due to anomalous diffusion, is used in modeling the ion electro diffusion at the neurons; the time-fractional modified anomalous sub diffusion equation, which is derived from the neural cell adhesion molecules, is usedfor describing processes that become less anomalous as time progresses by the inclusion of a second fractional timederivative acting on the diffusion term; the time fractional reaction sub diffusion equation is used to describe many different areas of chemical reactions, such as exciton quenching, recombination of charge carriers or radiation defects in solids, and predator pray relationships in ecology; the time-fractional Fokker–Planck equation is used to describe many phenomena in plasma and polymer physics, population dynamics, neurosciences, nonlinear hydrodynamics, pattern formation, and psychology; the time-fractional Fisher’s equation is used to describe the population growth models, whilst, the time fractional Newell–Whitehead equation is used to describe fluid dynamics model and capillary–gravity waves. The fractional differential equations, generalization of the classical mass-spring-damper models, are useful to understand the behavior of dynamical complex systems, mechanical vibrations, control theory, relaxation phenomena, viscoelasticity, viscoelastic damping and oscillatory processes (Gomez-Aguilaret al., 2015).The space-time fractional diffusion equations on two time intervals was used in finance to model option pricing and the model was shown to be useful for option pricing during some temporally abnormal periods (Korbel & Luchko, 2016). The α-fractional diffusion equation for $0<\alpha<2$ describes the so called Levy flights that correspond to the continuous time random walk model, where both the mean waiting time and the jump length variance of the diffusing Particles are divergent(Luchko, 2016). Time fractional diffusion equations in the Caputo sense with initial conditions are used to model cancer tumor(Iyiola & Zaman, 2014).

Nonlinear diffusion equations play a great role to describe the density dynamics in a material undergoing diffusion in a dynamic system which includes different branches of science and technology. The classical and simplest diffusion equation which is used to model the free motion of the particle is:

$$\frac{\partial u(x,t)}{\partial t} = A\frac{\partial^2}{\partial x^2}u(x,t) - \frac{\partial}{\partial x}\left(F(x)u(x,t)\right), \text{A} > 0 \qquad (1.1),$$

where $u(x,t)$ is the probability density function of finding a particle at the point x in time instant t, $F(x)$ is the external force, and A is a positive constant which depends on the temperature, the friction coefficient, the universal gas constant and the Avogadro number (A.Kumaret al., 2017).

Recently, the fractional differential equations have gained much attention of researchers due to the fact that they generate fractional Brownian motion which is generalization of Brownian motion(Podlubny, 1999).Das, Visha, Gupta and Saha Ray(2011) stated that time fractional diffusion equation, which is one of the fractional differential equations, is obtained from the classical diffusion equation in mathematical physics by replacing the first order time derivative by a fractional derivative of order α where $0<\alpha<1$. Time fractional diffusion equation is an evolution equation that generates the fractional Brownian motion (FBM) which is a generalization of Brownian motion (Das, et al., 2011; Podlubny, 1999). Due to the fact that fractional derivative provides an excellent tool for describing memory and hereditary properties for various materials and processes(Caputo & Mainardi, 1971), thetime fractional diffusion equations(A.Kumar et al., 2017; Cetinkaya & Kiymaz, 2013;Das, 2009;Kebede, 2018; Kumaret al., 2012) were extended to the form

$$\begin{cases} \dfrac{\partial^\beta u}{\partial t^\beta} = D\sum\limits_{i=1}^{n}\dfrac{\partial^2}{\partial x_i^2}u - \sum\limits_{i=1}^{n}\dfrac{\partial}{\partial x_i}[F(x)u]; \; 0<\beta<1,\; D>0, t>0,\; x = x_1, x_2, \cdots, x_n, x_1>0, x_2>0, \cdots, x_n>0 & (1.2a) \\ subject\ to\quad I.C:\; u^k(x,0) = f_k(x);\; k=0,1,2,\cdots, q-1;\; x = x_1, x_2, \cdots, x_n;\; x_1>0, x_2>0, \cdots, x_n>0 & (1.2b) \end{cases}$$

,which is generalization of equation (1.1),was considered in this study. Here,

$$D_t^\beta u(x,t) = J_t^{1-\beta}\left[\frac{\partial}{\partial t}u(x_1,x_2,\cdots x_n,t)\right], \text{and } u = u(x_1, x_2, \cdots, x_n, t).$$

The fractional derivative D_t^β is considered in the Caputo sense which has the main advantage that the initial conditions for fractional differential equations with Caputo derivative take on the same form as for integer order differential equations (Caputo, 1967). Due to this, considerable works on fractional diffusion equations have already been done by different authors to obtain exact, approximate analytic and pure numerical solutions by using various developed methods.

Recently, Adomian Decomposition Methodby Saha Ray and Bera in 2006 (As cited in Cetinkaya & Kiymaz, 2013; Kumar et al., 2012; Das, 2009),variational iteration method (Das, 2009), Homotopy Analysis Method(Das, et al., 2011), Laplace Transform Method (Kumaret al., 2012), Generalized Differential Transform

Method(Cetinkaya & Kiymaz, 2013)and Residue fractional power series method (Kumaret. al., 2017), fractional reduced differential transform method (kebede, 2018)which have their own inbuilt deficiencies: the complexity and difficulty of solution procedure for calculation of adomain polynomials, the restrictions on the order of the nonlinearity term or even the form of the boundary conditions and uncontrollability of non-zero end conditions, unrestricted freedom to choose base function to approximate the linear and nonlinear problems, and complex computations respectively, were used to obtain solutions of (1+1) dimensional time fractional diffusion equations with initial conditions. To overcome these deficiencies, the iterative fractional Laplace transform method(IFLTM)was preferably taken in this paper to solve (n+1) dimensional time fractional diffusion equations with initial conditions of the form $(1.2a)$ given that $(1.2b)$ analytically.

The iterative method was firstly introduced by Daftardar-Gejji and Jafari(2006) to solve numerically the nonlinear functional equations. By now, the iterative method has been used to solve many integer and fractional boundary value problem (Daftardar-Gejji & Bhalekar, 2010). Jafari et al.(2013) firstly solved the fractional partial differential equations by the use of iterative Laplace transform method (ILTM). More recently, Yan (2013),Sharma and Bairwa (2015), and Sharma and Bairwa(2014)were used ILTM for solving Fractional Fokker-Planck equations, generalized time-fractional biological population model, and Fractional Heat and Wave-Like Equations respectively.

In this paper, the author has been examined how to obtain the solutions of (n+1) dimensional time fractional diffusion equations with initial conditions in the form infinite fractional power series, in terms of Mittage Lefler function of one parameter and exact form by the use of iterative fractional Laplace transform method (IFLTM). The basic idea of IFLTM was developed successfully. To see its effectiveness and applicability, three test examples were presented. Their closed solutions in the form of infinite fractional power series and in terms of Mittag-Leffler functions in one parameter, which rapidly converge to exact solutions, were successfully derived by the use iterative fractional Laplace transform method (IFLTM). The results show that the iterative fractional Laplace transform method works successfully in solving (n+1) dimensional time fractional diffusion equations in a direct way without using linearization, perturbation, discretization or restrictive assumptions, and hence it can be extended to other fractional differential equations.

This paper is organized as follows: in the next sections which is the methodology, which is the way the study was designed to go through, was discussed. In section 3, results and discussion which include: some definitions, properties and theorems of fractional calculus theory, the results which are is the basic idea of fractional Laplace transform method, application models and discussion of application of the results obtained were presented. Finally, conclusions are presented in Section 4.

METHODOLOGY

In this paper, it was designed to set the theoretical foundation of the study to come to its objective. Next, it was designed to consider time fractional differential equations under initial conditions, which are(n+1) dimensional time fractional diffusion equations with initial conditions of the form(1.2a) given that(1.2b) and then use analytical design to solve the manalytically by using iterative fractional Laplace transform method by following the next five procedures sequentially. First, it was designed to revisit some basic definitions and properties of fractional calculus and Laplace transform. Secondly, it was designed to develop basic idea of iterative fractional Laplace transform method for(3.10a) given that (3.10b) and then obtain a remark 3.2.2.1. Thirdly, it was designed to obtain closed solutions of (1.2a) given that (1.2b) in the form of infinite fractional power series by using the remark 3.2.2.1. Fourthly, it was designed to determine closed solutions equations of the form of (1.2a) given that (1.2b) in terms of Mittag-Leffler functions in one parameter from these infinite fractional power series. Lastly, it was designed to obtain exact solutions of (1.2a) given that (1.2b) for the special case $\alpha = 1$.

RESULTS AND DISCUSSION

Preliminaries and Notations

Fractional Calculus

Here, some basic definitions and properties of fractional calculus and Laplace transform were revisited as follows to use them in this paper;

see (Kilbas,Srivastava, & Trujillo, 2006; Mainardi, 2010; Podlubny, 1999; Millar & Ross, 1993).

Definition3.1.1. A real valued function $u(x,t), x \in IR, t > 0$, is said to be in the space $C_\mu, \mu \in IR$, if there exists a real number $q > \mu$ such that $u(x) = t^q u_1(x,t)$, where $u_1(x,t) \in C(IR \times [0,+\infty))$ and it is said to be in the space C_μ^m if $u^{(m)}(x,t) \in C_\mu, n \in IN$.

Definition3.1.2.The Riemann-Liouville fractional integral operator of order $\beta \geq 0$ of a function $u(x,t) \in C_\mu, \mu > -1$ is defined as

$$J_t^\beta u(x,t) = \begin{cases} \dfrac{1}{\Gamma(\beta)} \displaystyle\int_0^t (x-\xi)^{\beta-1} u(x,\xi) d\xi, \ 0 < \xi < t, \beta > 0 \\ u(x,t), \beta = 0 \end{cases} \tag{3.1}$$

Consequently, for $\alpha, \beta \geq 0, C \in IR, u(x,t) \in C_\mu^m, u(x,t) \in C_\mu, \mu > -1$, the operator J_t^β has the following properties:

$$J_t^\alpha J_t^\beta u(x,t) = J_t^{\alpha+\beta} u(x,t) = J_t^\beta J_t^\alpha u(x,t)$$

$$J_t^\alpha c = \left(\frac{c}{\Gamma(\alpha+1)} \right) t^\alpha.$$

$$J_t^\alpha t^\gamma = \left(\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)} \right) t^{\gamma+\alpha}.$$

The Riemann Liouville derivative has the disadvantage that it does not allow the utilization of initial and boundary conditions involving integer order derivatives when trying to model real world problems with fractional differential equations. To beat this disadvantage of Riemann Liouville derivative (Millar & Ross, 1993; Podlubny, 1999), Caputo proposed a modified fractional differentiation operator D_a^β (Caputo & Mainardi, 1971)to illustrate the theory of viscoelasticity as follows:

$$D_a^\beta f(x) = J_a^{m-\beta} D^m f(x) = \frac{1}{\Gamma(m-\beta)} \int_a^x (x-\xi)^{m-\beta-1} f^{(m)}(\xi) d\xi, \beta \geq 0 \tag{3.2}$$

, where $m-1 < \beta < m, x > a$ and $f \in C_{-1}^m$.

This Caputo fractional derivative allows the utilization of initial and boundary conditions involving integer order derivatives, which have clear physical interpretations of the real situations.

Definition3.1.3. For the smallest integer that exceeds β ,the Caputo time fractional derivative order $\beta > 0$ of a function $u(x,t)$ is defined as:

$$D_t^\beta u(x,t) = \begin{cases} \dfrac{1}{\Gamma(m-\beta)} \displaystyle\int_0^t (t-\xi)^{m-\beta-1} \frac{\partial^m u(x,\xi)}{\partial \xi^m} d\xi \\ \dfrac{\partial^m u(x,t)}{\partial t^m}, \beta = m \end{cases} = J^{m-\beta} \frac{d^m}{dt^m} u(x,t), \ 0 \leq m-1 < \beta < m \tag{3.3}$$

Theorem3.1.1.If $m-1 < \beta \leq m, \forall m \in IN, u(x,t) \in C_\gamma^m, \gamma \geq -1$ then

$$D_t^\beta J_t^\beta u(x,t) = u(x,t).$$

$$J^\beta D^\beta u(x,t) = u(x,t) - \sum_{k=0}^{m-1} \frac{\partial^k}{\partial t^k} u(x,0^+) \frac{t^k}{k!}, \ t > 0.$$

The reader is kindly requested to go through (Kilbas, Srivastava, & Trujillo, 2006; Mainardi, 2010) in order to know more details about the mathematical properties of fractional derivatives and fractional integrals, including their types and history, their motivation for use, their characteristics, and their applications.

Definition3.1.4.According toMillar and Ross(1993), Podlubny(1999), and Sontakke and Shaikh(2015), the Mittag-Leffler function, which is a one parameter generalization of exponential function, is defined as

$$E_{\alpha}(z)=\sum_{q=0}^{\infty}\frac{z^{q}}{\Gamma(q\alpha+1)},\alpha\in C,\mathrm{Re}(\alpha)>0 \tag{3.4}$$

Definition3.1.5. (Kilbas, Srivastava, & Trujillo, 2006) Laplace transform of $\phi(t),t>0$ is

$$L[f(t)]=F(s)=\int_{0}^{\infty}e^{-st}f(t)dt \tag{3.5}$$

Definition3.1.6. (Kilbas, Srivastava, & Trujillo, 2006)Laplace transform of $D_t^{\beta}u(x,t)$ is

$$L[D_t^{\beta}u(x,t)]=L[u(x,t)]-\sum_{k=1}^{q-1}u^{k}(x,0)s^{\beta-k-1},q-1<\beta\leq q,q\in IN \tag{3.6}$$

Main Results

Some Basic Definitions of Fractional Calculus and Laplace Transform

Here, some definitions of fractional calculus and Laplace transform, one theorem and basic idea of iterative fractional Laplace transform method were developed and introduced.

Definition3.2.1 A real valued $(n+1)$ dimensional function $u(x_1,x_2,\cdots x_n,t)$, where $(x_1,x_2,\cdots x_n)\in IR^{n},t>0$ is said to be in the space C_{μ}, $\mu\in IR$,if there exists a real number $p>\mu$ such that $u(x)=t^{p}u_1(x_1,x_2,\cdots x_n,t)$, where $u_1(x_1,x_2,\cdots x_n,t)\in C(IR^{n}\times[0,+\infty))$ and it is said to be in the space C_{μ}^{m} if $u^{(m)}(x_1,x_2,\cdots x_n,t)\in C_{\mu},n\in IN$.

Definition3.2.2.The Riemann-Liouville fractional integral operator of order $\beta\geq 0$ of $(n+1)$dimensional function $u(x_1,x_2,\cdots x_n,t)\in C_{\mu},\mu>-1$is defined as

$$J_t^{\beta}u(x_1,x_2,\cdots x_n,t)=\begin{cases}\dfrac{1}{\Gamma(\beta)}\displaystyle\int_0^t(x-\xi)^{\beta-1}u(x_1,x_2,\cdots x_n,\xi)d\xi, & 0<\xi<\mathrm{t},\beta>0\\ u(x_1,x_2,\cdots x_n,t), & \beta=0\end{cases} \tag{3.7}$$

Lemma3.2.1.For $\alpha,\beta\geq 0,C\in IR,u(x_1,x_2,\cdots x_n,t)\in C_{\mu}^{m},u(x_1,x_2,\cdots x_n,t)\in C_{\mu},\mu>-1,$ the operator J_t^{β} has the property:

$$J_t^{\alpha}J_t^{\beta}u(x_1,x_2,\cdots x_n,t)=J_t^{\alpha+\beta}u(x_1,x_2,\cdots x_n,t)=J_t^{\beta}J_t^{\alpha}u(x_1,x_2,\cdots x_n,t)$$

Definition3.2.3. For the smallest integer that exceeds β ,the Caputo time fractional derivative order $\beta>0$ of $(n+1)$dimensional function, $u(x_1,x_2,\cdots x_n,t)$ is defined as:

$$D_t^{\beta}u(x_1,x_2,\cdots x_n,t)=\begin{cases}\dfrac{1}{\Gamma(m-\beta)}\displaystyle\int_0^t(t-\xi)^{m-\beta-1}\frac{\partial^{m}u(x_1,x_2,\cdots x_n,\xi)}{\partial\xi^{m}}d\xi\\ \dfrac{\partial^{m}u(x_1,x_2,\cdots x_n,t)}{\partial t^{m}}, \beta=m\end{cases}=J^{m-\beta}\frac{d^{m}}{dt^{m}}u(x_1,x_2,\cdots x_n,t),\ 0\leq m-1<\beta<m \tag{3.8}$$

Theorem3.2.1.If $m-1<\beta\leq m,\forall\mathrm{m}\in\mathrm{IN}$,

$(n+1)$dimensional function $u(x_1,x_2,\cdots x_n,t)\in C_{\gamma}^{m},\gamma\geq -1$, then

$$D_t^{\beta}J_t^{\beta}u(x_1,x_2,\cdots x_n,t)=u(x_1,x_2,\cdots x_n,t).$$

$$J^{\beta}D^{\beta}u(x_1,x_2,\cdots x_n,t)=u(x_1,x_2,\cdots x_n,t)-\sum_{k=0}^{m-1}\frac{\partial^k}{\partial t^k}u(x_1,x_2,\cdots x_n,0^+)\frac{t^k}{k!}, \; t>0.$$

Definition3.2.4. Laplace transform of $D_t^{\beta}u(x,t)$ is

$$L[D_t^{\beta}u(x_1,x_2,\cdots x_n,t)]=L[u(x_1,x_2,\cdots x_n,t)]-\sum_{r=0}^{p-1}u^r(x_1,x_2,\cdots x_n,0)s^{\beta-r-1}, p-1<\beta\le p, p\in IN \quad (3.9)$$

where $u(x_1,x_2,\cdots x_n,t)$ is $(n+1)$dimensional function and $u(x_1,x_2,\cdots x_n,0)$ is the r order derivative of $u(x_1,x_2,\cdots x_n,t)$ at $t=0$.

3.2.2. Basic idea ofIterative fractional Laplace transform method

The basic idea of this method is illustrated as follows.

Consider a general $(n+1)$ dimensional time fractional non-linear non homogeneous partial differential equation with initial conditions of the form:

$$\begin{cases} D_t^{\beta}u+Lu+Nu=f;\ p-1<\beta\le p & (3.10a)\\ u_0^r=g_r;\ r=0,1,2,\cdots,p-1 & (3.10b)\end{cases}$$

, where $u=u(x_1,x_2,\cdots,x_n,t)$, $u_0^r=g_r(x_1,x_2,\cdots,x_n,0)$, $D_t^{\beta}u(x_1,x_2,\cdots,x_n,t)$ is the Caputo fractional derivative of the function, , L is the linear operator, N is general nonlinear operator and $f(x_1,x_2,\cdots,x_n,t)$ is the source term respectively.

Now apply fractional Laplace transform method to $(3.10a)$given that$(3.10b)$.as follows.

Applying the Laplace transform denoted by L in equation $(3.10a)$, we obtain:

$$L[D_t^{\beta}u]+L[Ru+Nu]=L[f] \quad (3.11)$$

By using equation (3.9), we get:

$$L[u]=\frac{1}{s^{\beta}}\sum_{r=0}^{p-1}u^r s^{\beta-r-1}+\frac{1}{s^{\beta}}L[f]-\frac{1}{S^{\beta}}L[Ru+Nu] \quad (3.12)$$

Taking inverse Laplace transform of equation(3.12), we get:

$$u=L^{-1}\left[\frac{1}{S^{\beta}}\left[\sum_{r=0}^{r-1}u^r s^{\beta-r-1}+L[f]\right]\right]-L^{-1}\left[\frac{1}{S^{\beta}}L[Ru+Nu]\right] \quad (3.13)$$

Now we apply the iterative methodto(3.13)as follows.

Let u be the solution of $(3.10a)$ and has the infinite series form

$$u=\sum_{i=0}^{\infty}u_i \quad (3.14)$$

Since, R is the linear operator, using equation (3.14),

$$Ru=R\sum_{i=0}^{\infty}u_i=\sum_{i=0}^{\infty}Ru_i \quad (3.15)$$

Since N is the non-linear operator, by using equation (3.14), N is decomposed as:

$$Nu = N\left(\sum_{i=1}^{\infty} u_i\right) = N(u_0) + \sum_{i=0}^{\infty}\left(N\left(\sum_{r=0}^{i} u_r\right) - N\left(\sum_{r=0}^{i-1} u_r\right)\right) \quad (3.16)$$

By substituting Equations (3.14), (3.15) and (3.16) in Equation (3.13), we get

$$\sum_{i=1}^{\infty} u_i = L^{-1}\left[\frac{1}{S^{\beta}}\left[\sum_{r=0}^{p-1} u^{r} s^{\beta-r-1} + L[f]\right]\right] - L^{-1}\left[\frac{1}{S^{\beta}} L\left[\sum_{i=0}^{\infty} Ru_i + N(u_0) + \sum_{i=1}^{\infty}\left(N\left(\sum_{r=0}^{i} u_j\right) - N\left(\sum_{r=0}^{i-1} u_r\right)\right)\right]\right] \quad (3.17)$$

Now from Equation (3.17), we define recurrence relations as follows:

$$u_0 = L^{-1}\left[\frac{1}{S^{\beta}}\left[\sum_{r=0}^{p-1} u^{r} s^{\beta-r-1} + L[f]\right]\right] \quad (3.18)$$

$$u_1 = -L^{-1}\left[\frac{1}{S^{\beta}} L[Ru_0 + N(u_0)]\right] = -L^{-1}\left[\frac{1}{S^{\beta}} L[R(u_0) + N(u_0)]\right] \quad (3.19)$$

$$u_2 = u_{1+1} = -L^{-1}\left[\frac{1}{S^{\beta}} L[Ru_1 + N(u_0+u_1) - N(u_0)]\right] = -L^{-1}\left[\frac{1}{S^{\beta}} L\left[R\left[\sum_{i=0}^{1} u_i - u_0\right] + N\left(\sum_{i=0}^{1} u_i\right) - N(u_0)\right]\right] \quad (3.20)$$

$$u_3 = u_{2+1} = -L^{-1}\left[\frac{1}{S^{\beta}} L[Ru_2 + N(u_0+u_1+u_2) - N(u_0+u_1)]\right] = -L^{-1}\left[\frac{1}{S^{\beta}} L\left[R\left(\sum_{i=0}^{2} u_i - \sum_{i=0}^{1} u_i\right) + N\left(\sum_{i=0}^{2} u_i\right) - N\left(\sum_{i=0}^{1} u_i\right)\right]\right] \quad (3.21)$$

Continuing with this procedure, we get

$$u_i = u_{p+1} = -L^{-1}\left[\frac{1}{S^{\beta}} L\left[R\left(\sum_{i=0}^{p} u_i - \sum_{i=0}^{p-1} u_i\right) + N\left(\sum_{i=0}^{p} u_i\right) - N\left(\sum_{i=0}^{p-1} u_i\right)\right]\right];\ p \in IN,\ p \geq 1,\ i = 0, 1, 2, \cdots, p+1 \quad (3.22)$$

Therefore the i^{th} term approximate solution of Equation $(3.10a)$ given that $(3.10b)$ in series form is given by

$$\tilde{u}_i \cong u_0 + u_1 + u_2 + \cdots + u_{p+1};\ p = 1, 2, 3, \cdots \quad (3.23)$$

The infinite power series form solution of $(3.10a)$ given that $(3.10b)$ as $p \in IN$ approaches ∞, is obtained fromEquation (3.23) and it is given as Equation (3.14).

The solution of $(3.10a)$ given that $(3.10b)$ in term of Mittag Leffler function of one parameter is obtained from step5.

Remark3.2.2.1: If $Lu = -\left(D\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} u - \sum_{i=1}^{n}\frac{\partial}{\partial x_i}[F(x)u]\right)$, $Nu = 0$ and $f = 0$, then Equation $(3.10a)$ given that $(3.10b)$ becomes Equation $(1.2a)$ given that $(1.2b)$ and

u_0 which is given by Equation (3.18) becomes

$$u_0 = L^{-1}\left[\frac{1}{S^{\beta}}\left[\sum_{r=0}^{p-1} u^{r} s^{\beta-r-1}\right]\right] \quad (3.24)$$

u_1

which is given by Equation (3.19) becomes

$$u_1 = L^{-1}\left[\frac{1}{S^{\beta}}L\left[\left(D\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}u_0 - \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left[F(x)u_0\right]\right)\right]\right] \quad (3.25)$$

$u_i = u_{p+1}$ which is given by Equation (3.20) becomes

$$u_i = u_{p+1} = L^{-1}\left[\frac{1}{S^{\beta}}L\left[D\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}\left(\sum_{i=0}^{p}u_i - \sum_{i=0}^{p-1}u_i\right) - \sum_{i=1}^{n}\frac{\partial}{\partial x_i}F(x)\left(\sum_{i=0}^{p}u_i - \sum_{i=0}^{p-1}u_i\right)\right]\right];\ p \in IN, p \geq 1, i = 0, 1, 2, \cdots, p+1 \quad (3.26)$$

Applications

To validate (show) the simplicity, effectiveness and applicability of iterative fractional Laplace transform method (IFLTM) for determining closed solutions of (n+1) dimensional time fractional diffusion equations of the form $(1.2a)$ given that $(1.2b)$ in infinite fractional power series form, in terms of Mittag-Leffler functions in one parameter and exact form, threeapplication examples were considered and solved as follows.

Example3.2.1.1.Taking $F(x_1) = -x_1, \lambda = 1$ in $(1.2a)$ and choosing $f(x_1) = 1$ in $(1.2b)$ (A.Kumar et al., 2017; Cetinkaya & Kiymaz, 2013; Kebede, 2018; Kumar et al., 2012),consider the initial value problem:

$$\begin{cases}\dfrac{\partial^{\beta}u}{\partial t^{\beta}} = \dfrac{\partial^2 u}{\partial x_1^{\,2}} + \dfrac{\partial}{\partial x_1}(x_1 u),\ x_1 > 0,\ t > 0,\ 0 < \beta \leq 1\\ \text{Subject to initial condition}: u(x_1, 0, \cdots, 0, 0) = 1\end{cases}$$

Since $F(x_1) = -x_1, \lambda = 1$ and $f(x_1) = 1$,

By Equation (3.24):

$$u_0(x_1, t) = L^{-1}\left[\frac{1}{S^{\beta}}\left[u^{0}(x_1, 0)s^{\beta-0-1}\right]\right] = L^{-1}\left[\frac{1}{S^{\beta}}\left[u^{0}(x_1, 0)s^{\beta-0-1}\right]\right] = L^{-1}\left[\frac{1}{S^{\beta}}\times 1 \times s^{\beta-1}\right] = 1$$

$$u_0(x_1, t) = u_0(x_1, t) = 1 \quad (3.28)$$

By Equation (3.25):

$$u_1(x_1, t) = L^{-1}\left[\frac{1}{S^{\beta}}L\left[\left(1\times\frac{\partial^2}{\partial x_1^2}(1) - \frac{\partial}{\partial x_1}\left[(-x_1)\times 1\right]\right)\right]\right] = L^{-1}\left[\frac{1}{S^{\beta}}\right] = \frac{t^{\beta}}{\Gamma(\beta+1)}$$

$$u_1(x_1, t) = \frac{t^{\beta}}{\Gamma(\beta+1)},\quad 0 < \beta \leq 1, x > 0, t > 0 \quad (3.29)$$

By Equation (3.26):

For $p = 1$, $u_2 = L^{-1}\left[\frac{1}{S^{\beta}}L\left[D\frac{\partial^2}{\partial x_1^2}(u_0 + u_1 - u_0) - \frac{\partial}{\partial x_1}F(x_1)(u_0 + u_1 - u_0)\right]\right]$

$$u_2(x_1, t) = L^{-1}\left[\frac{1}{S^{\beta}}L\left[\left(1\times\frac{\partial^2}{\partial x_1^2}\left(1 + \frac{t^{\beta}}{\Gamma(\beta+1)} - 1\right) - \frac{\partial}{\partial x_1}\left[(-x_1)\times\left(1 + \frac{t^{\beta}}{\Gamma(\beta+1)} - 1\right)\right]\right)\right]\right] = L^{-1}\left[\frac{1}{S^{\beta}}L\left[\frac{t^{\beta}}{\Gamma(\beta+1)}\right]\right] = L^{-1}\left[\frac{1}{S^{2\beta}}\right]$$

$$u_2(x_1, t) = \frac{t^{2\beta}}{\Gamma(2\beta+1)},\quad 0 < \beta \leq 1, x > 0, t > 0 \quad (3.30)$$

For $p = 2$, $u_3 = L^{-1}\left[\frac{1}{S^{\beta}}L\left[D\frac{\partial^2}{\partial x_1^2}\left(u_0 + u_1 + u_2 - (u_0 + u_1)\right) - \frac{\partial}{\partial x_1}F(x_1)\left(u_0 + u_1 + u_2 - (u_0 + u_1)\right)\right]\right]$

$$u_3(x_1,t)=L^{-1}\left[\frac{1}{S^{\beta}}L\left[\left(1\times\frac{\partial^2}{\partial x_1^2}\left(1+\frac{t^{\beta}}{\Gamma(\beta+1)}+\frac{t^{2\beta}}{\Gamma(2\beta+1)}-\left(1+\frac{t^{\beta}}{\Gamma(\beta+1)}\right)\right)-\frac{\partial}{\partial x_1}\left[(-x_1)\times\left(1+\frac{t^{\beta}}{\Gamma(\beta+1)}+\frac{t^{2\beta}}{\Gamma(2\beta+1)}-\left(1+\frac{t^{\beta}}{\Gamma(\beta+1)}\right)\right)\right]\right)\right]\right]$$

$$u_3(x_1,t)=L^{-1}\left[\frac{1}{S^{\beta}}L\left[\frac{t^{2\beta}}{\Gamma(2\beta+1)}\right]\right]=L^{-1}\left[\frac{1}{S^{\beta}}\times\frac{1}{S^{2\beta}}\right]=L^{-1}\left[\frac{1}{S^{3\beta}}\right]=\frac{t^{3\beta}}{\Gamma(3\beta+1)}$$

$$u_3(x_1,t)=\frac{t^{3\beta}}{\Gamma(3\beta+1)},\quad 0<\beta\le 1, x>0, t>0 \tag{3.31}$$

Continuing with this process, we obtain that:

$$u_i=u_{p+1}=u_1(x_1,t)=\frac{t^{i\beta}}{\Gamma(i\beta+1)},\quad 0<\beta\le 1, x>0, t>0, i=1,\ 2,\ 3,\ \cdots,P+1, P\in IN \tag{3.32}$$

The i^{th} order approximate solution of Equation $(3.27a)$ given that $(3.27b)$, denoted by $\tilde{u}_i(x_1,t)$ is given by:

$$\tilde{u}_i(x_1,t)=\sum_{i=0}^{p+1}\left(\frac{1}{\Gamma(i\beta+1)}\right)t^{i\beta}, 0<\beta\le 1, x>0, t>0 \tag{3.33}$$

By letting $p\in IN$ to ∞ or taking limit of both sides of Equation (3.33) as $p\in IN\to\infty$, the closed solution of Equation (3.27a) in the form of infinitefractional power seriesdenoted by $u(x_1,t)$ is:

$$u(x_1,t)=\sum_{i=0}^{\infty}\frac{t^{i\beta}}{\Gamma(i\beta+1)}, 0<\beta\le 1, x>0, t>0 \tag{3.34}$$

Thus, by using Equation (3.4) in Equation (3.34), the closed solution of Equation (3.23a) in terms of Mittag-Leffler functionof one parameter is given by:

$$u(x_1,t)=E_{\beta}\left(t^{\beta}\right), 0<\beta\le 1, x>0, t>0 \tag{3.35}$$

If $\beta=\frac{1}{2}$, then Equation (3.35) becomes $u(x_1,t)=E_{\frac{1}{2}}\left(\sqrt{t}\right)$

Lastly, the exact solution of Equation (3.27a), $u_{exact}(x_1,t)$ can be obtained from Equation (3.27) as β approaches to 1 from left and it is given by

$$u_{exact}(x_1,t)=e^{t}, \beta=1, x>0, t>0 \tag{3.36}$$

In order to guarantee the agreement between the exactsolution, Equation (3.36) and the i^{th} order approximate solution, Equation (3.33) of Equation (3.27a) given that Equation (3.27b), the absolute errors: $E_5(u)=\left|u_{exact}(x_1,t)-\tilde{u}_5(x_1,t)\right|$ and $E_6(u)=\left|u_{exact}(x_1,t)-\tilde{u}_6(x_1,t)\right|$ were computed as they were shown below by tables 3.1and 3.2by considering the 5^{th} order approximate solution, $\tilde{u}_5(x_1,t)=\sum_{i=0}^{5}\frac{t^{i\beta}}{\Gamma(i\beta+1)}, x_1, t, \beta\in\{0.25, 0.5, 0.75, 1\}$ and the 6^{th} order approximate solutions, $\tilde{u}_6(x_1,t)=\sum_{k=0}^{6}\frac{t^{i\beta}}{\Gamma(i\beta+1)}, x_1, t, \beta\in\{0.25, 0.5, 0.75, 1\}$ of Equation (3.27a) given that Equation

$(3.27b)$ without loss of generality.

Table3.1. Absolute error of approximating the solution ofEquation(3.27a) given that Equation(3.27b) to5thorderusing IFLTM

Variables		Absolute error, $E_5(u)=\lvert u_{exact}(x_1,t)-\tilde{u}_5(x_1,t)\rvert$			
t	x_1	$\alpha=0.25$	$\alpha=0.50$	$\alpha=0.75$	$\alpha=1$
0.25	0.25	4.852591	0.827357	0.530995	3.515836×10^{-7}
0.25	0.50	4.852591	0.827357	0.530995	3.515836×10^{-7}
0.25	0.75	4.852591	0.827357	0.530995	3.515836×10^{-7}
0.25	1.00	4.852591	0.827357	0.530995	3.515836×10^{-7}
0.50	0.25	6.317463	1.770632	0.449889	2.335403×10^{-5}
0.50	0.50	6.317463	1.770632	0.449889	2.335403×10^{-5}
0.50	0.75	6.317463	1.770632	0.449889	2.335403×10^{-5}
0.50	1.00	6.317463	1.770632	0.449889	2.335403×10^{-5}
0.75	0.25	7.356235	2.253027	0.544978	0.000276
0.75	0.50	7.356235	2.253027	0.544978	0.000276
0.75	0.75	7.356235	2.253027	0.544978	0.000276
0.75	1.00	7.356235	2.253027	0.544978	0.000276
1.00	0.25	8.108369	2.660339	0.591061	0.001615
1.00	0.50	8.108369	2.660339	0.591061	0.001615
1.00	0.75	8.108369	2.660339	0.591061	0.001615
1.00	1.00	8.108369	2.660339	0.591061	0.001615

Table 3.2. Absolute error of approximating the solution of Equation(3.27a) given that Equation (3.27b) to6thorderusing IFLTM

Variables		Absolute error, $E_6(u)=\lvert u_{exact}(x_1,t)-\tilde{u}_6(x_1,t)\rvert$			
t	x_1	$\alpha=0.25$	$\alpha=0.50$	$\alpha=0.75$	$\alpha=1$
0.25	0.25	4.915278	1.194657	0.047116	1.249937×10^{-8}
0.25	0.50	4.915278	1.194657	0.047116	1.249937×10^{-8}
0.25	0.75	4.915278	1.194657	0.047116	1.249937×10^{-8}
0.25	1.00	4.915278	1.194657	0.047116	1.249937×10^{-8}
0.50	0.25	6.494771	1.791466	0.087453	1.652645×10^{-6}
0.50	0.50	6.494771	1.791466	0.087453	1.652645×10^{-6}
0.50	0.75	6.494771	1.791466	0.087453	1.652645×10^{-6}
0.50	1.00	6.494771	1.791466	0.087453	1.652645×10^{-6}
0.75	0.25	7.681970	2.32334	0.126940	2.919142×10^{-5}
0.75	0.50	7.681970	2.32334	0.126940	2.919142×10^{-5}
0.75	0.75	7.681970	2.32334	0.126940	2.919142×10^{-5}
0.75	1.00	7.681970	2.32334	0.126940	2.919142×10^{-5}
1.00	0.25	8.609871	2.827005	0.595306	0.000226
1.00	0.50	8.609871	2.827005	0.595306	0.000226
1.00	0.75	8.609871	2.827005	0.595306	0.000226

1.00	1.00	8.609871	2.827005	0.595306	0.000226

Example3.2.3.2.Taking $F(x_1,x_2)=-x_1-x_2, \lambda=1,\ u=u(x_1,x_2,\ t)$ and choosing $f(x_1,x_2)=x_1+x_2$,in Equation $(1.2a)$, consider the initial value problem:

$$\begin{cases}\dfrac{\partial^\beta u}{\partial t^\beta}=\dfrac{\partial^2 u}{\partial x_1{}^2}+\dfrac{\partial^2 u}{\partial x_2{}^2}+\left(\dfrac{\partial}{\partial x_1}+\dfrac{\partial}{\partial x_2}\right)((x_1+x_2)u), x_1>0, x_2>0, t>0, 0<\beta\le 1 & (3.37a)\\ \text{Subject to initial condition } u(x_1,x_2,0)=x_1+x_2 & (3.37b)\end{cases}$$

Since $F(x_1,x_2)=-x_1-x_2, \lambda=1$ and $f(x_1,x_2)=x_1+x_2$

By Equation (3.24):

$$u_0(x_1,x_2,\ t)=L^{-1}\left[\frac{1}{S^\beta}\left[u^{\,0}(x_1,x_2,0)s^{\beta-0-1}\right]\right]=\ L^{-1}\left[\frac{1}{S^\beta}\left[(x_1+x_2)s^{\beta-0-1}\right]\right]=\ L^{-1}\left[\frac{1}{S}\times(x_1+x_2)\right]=x_1+x_2$$

$$u_0(x_1,x_2,\ t)=u_0(x_1,x_2,\ t)=x_1+x_2 \qquad (3.38\ )$$

By Equation (3.25):

$$u_1((x_1,x_2,\ t)=L^{-1}\left[\frac{1}{S^\beta}L\left[\left(1\times\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)(x_1+x_2)\ -\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_1}\right)\left[(-x_1-x_2)\times(x_1+x_2)\right]\right)\right]\right]$$

$$u_1((x_1,x_2,\ t)=L^{-1}\left[\frac{1}{S^\beta}L[3x_1+3x_2\ ]\right]=L^{-1}\left[\frac{1}{S^\beta}\times\frac{1}{s}\times[3x_1+3x_2\ ]\right]=\frac{3(x_1+x_2)t^\beta}{\Gamma(\beta+1)}$$

$$u_1(x_1,x_2,t)=\frac{3(x_1+x_2)t^\beta}{\Gamma(\beta+1)},\quad 0<\beta\le 1, x_1>0, x_2>0, t>0 \qquad (3.39)$$

By Equation (3.26):

For $p=1$, $u_2=\ L^{-1}\left[\frac{1}{S^\beta}L\left[D\frac{\partial^2}{\partial x_1^2}(u_0+u_1-u_0)-\frac{\partial}{\partial x_1}F(x_1)(u_0+u_1-u_0)\ \right]\right]$

$$u_2(x_1,x_2,t)=L^{-1}\left[\frac{1}{S^\beta}L\left[\left(1\times\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)\left(x_1+x_2+\frac{3(x_1+x_2)t^\beta}{\Gamma(\beta+1)}-(x_1+x_2)\right)\ -\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\right)\left[(-x_1-x_{21})\times\left(x_1+x_2+\frac{3(x_1+x_2)t^\beta}{\Gamma(\beta+1)}-(x_1+x_2)\right)\right]\right)\right]\right]$$

$$u_2(x_1,x_2,t)=L^{-1}\left[\frac{1}{S^\beta}L\left[\frac{3^2\,(x_1+x_2)t^\beta}{\Gamma(\beta+1)}\right]\right]=L^{-1}\left[\frac{1}{S^\beta}\times\frac{3^2(x_1+x_2)}{s^{\beta+1}}\right]=\frac{3^2\,(x_1+x_2)t^{2\beta}}{\Gamma(2\beta+1)}$$

$$u_2(x_1,x_2,t)=\frac{3^2(x_1+x_2)t^{2\beta}}{\Gamma(2\beta+1)},\quad 0<\beta\le 1, x_1>0, x_2>0, t>0 \qquad (3.40)$$

Continuing with this process, we obtain that:

$$u_i(x_1,x_2,t)=u_{p+1}(x_1,x_2,t)=\frac{3^i(x_1+x_2)t^{i\beta}}{\Gamma(i\beta+1)},\quad 0<\beta\le 1,\ x_1>0,\ x_2>0,\ t>0, i=1,\ 2,\ 3,\ \cdots,P+1, P\in IN \qquad (3.41)$$

.

The i^{th} order approximate solution of Equation $(3.37a)$ given that $(3.37b)$, denoted by $\tilde{u}_i(x_1, x_2, t)$ is given by:

$$\tilde{u}_i(x_1, x_2, t) = \sum_{i=0}^{p+1} \frac{3^i (x_1 + x_2) t^{i\beta}}{\Gamma(i\beta + 1)}, 0 < \beta \le 1, x_1 > 0, x_2 > 0, t > 0 \qquad (3.42)$$

By letting $p \in IN$ to ∞ or taking limit of both sides of Equation (3.33) as $p \in IN \to \infty$, the closed solution of Equation (3.27a) in the form of infinite fractional power series denoted by $u(x_1, x_2, t)$ is:

$$u(x_1, x_2, t) = \sum_{i=0}^{\infty} \frac{3^i (x_1 + x_2) t^{i\beta}}{\Gamma(i\beta + 1)}, 0 < \beta \le 1, x_1 > 0, x_2 > 0, t > 0 \qquad (3.43)$$

Thus, by using Equation (3.4) in Equation (3.43), the closed solution of Equation (3.37a) in terms of Mittag-Leffler function of one parameter is given by:

$$u(x_1, x_2, t) = (x_1 + x_2) E_\beta\left(3t^\beta\right), 0 < \beta \le 1, x_1 > 0, x_2 > 0, t > 0 \qquad (3.44)$$

Lastly, the exact solution of equation (3.37a), $u_{exact}(x_1, x_2, t)$ can be obtained from Equation (3.27) as β approaches to 1 from left and it is given by

$$u_{exact}(x_1, x_2, t) = (x_1 + x_2) e^{3t}, \beta = 1, x_1 > 0, x_2 > 0, t > 0 \qquad (3.45)$$

In order to show the agreement between the exact solution, equation (3.45) and the i^{th} order approximate solution, equation (3.42) of equation (3.37a) given that equation $(3.37b)$, the absolute errors: $E_4(u) = \left|u_{exact}(x_1, x_2, t) - \tilde{u}_4(x_1, x_2, t)\right|$ and $E_5(u) = \left|u_{exact}(x_1, x_2, t) - \tilde{u}_5(x_1, x_2, t)\right|$ were computed as they were shown below by tables 3.3 and 3.4 by considering the 4^{rd} order approximate solutions, $\tilde{u}_4(x_1, x_2, t) = \sum_{i=0}^{4} \frac{3^i (x_1 + x_2) t^{i\beta}}{\Gamma(i\beta + 1)}, x_1, x_2, t, \beta \in \left\{\frac{1}{3}, \frac{2}{3}, 1\right\}$ and the 5^{th} order approximate solutions, $\tilde{u}_5(x_1, x_2, t) = \sum_{i=0}^{5} \frac{3^i (x_1 + x_2) t^{i\beta}}{\Gamma(i\beta + 1)}, x_1, x_2, t, \beta \in \left\{\frac{1}{3}, \frac{2}{3}, 1\right\}$ of equation (3.37a) given that equation (3.37b) without loss of generality.

Table3.3. *Absolute error of approximating the solution of Equation(3.37a) given that Equation(3.37b) to4thorderusing IFLTM*

Variables			Absolute error, $E_4(u) = \left\|u_{exact}(x_1, x_2, t) - \tilde{u}_4(x_1, x_2, t)\right\|$		
t	x_1	x_2	$\frac{1}{3}$	$\frac{2}{3}$	1
$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	20.084832	2.803182	0.031324
$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	40.169667	5.606364	0.062648
$\frac{1}{3}$	1	1	60.254500	8.409546	0.093972

$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	41.182782	5.875512	0.654432
$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	82.365564	11.751027	1.308864
$\frac{2}{3}$	1	1	123.548346	17.626554	1.963296
1	$\frac{1}{3}$	$\frac{1}{3}$	59.516178	16.990026	2.473692
1	$\frac{2}{3}$	$\frac{2}{3}$	119.032359	33.980055	4.947384
1	1	1	178.548536	50.970084	7.421074

Table3.4. *Absolute error of approximating the solution of Equation(3.37a) given that Equation (3.37b) to5thorderusing IFLTM*

Variables			Absolute error, $E_5(u)=\left\|u_{exact}(x_1,x_2,t)-\tilde{u}_5(x_1,x_2,t)\right\|$		
t	x_1	x_2	$\frac{1}{3}$	$\frac{2}{3}$	1
$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	36.532842	3.25242	0.025768
$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	73.06569	6.504837	0.051536
$\frac{1}{3}$	1	1	109.598534	9.757256	0.077304
$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	97.535262	11.539632	0.476655
$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	195.070521	23.079261	0.95331
$\frac{2}{3}$	1	1	292.60578	34.618892	1.429965
1	$\frac{1}{3}$	$\frac{1}{3}$	167.187744	34.483632	1.123692
1	$\frac{2}{3}$	$\frac{2}{3}$	334.375491	68.967261	2.247384
1	1	1	501.563236	103.450892	3.371074

Example3.2.2.3.Taking $F(x_1,x_2,x_3)=e^{-x_1-x_2-x_3}$, $\lambda=1$, and $u=u(x_1,x_2,x_2,\ t)$ in $(1.2a)$ and choosing $f(x)=e^{x_1+x_2+x_3}$ in equation $(1.2b)$, we have the initial value problem:

$$\begin{cases}\dfrac{\partial^\beta u}{\partial t^\beta}=\dfrac{\partial^2 u}{\partial x^2}-\dfrac{\partial}{\partial x}(e^{-x_1-x_2-x_3}u),0<\beta\le 1, x>0, t>0 & (3.47a)\\ u(x,0)=e^{x_1+x_2+x_3} & (3.47b)\end{cases}$$

Since $F(x_1,x_2,x_3)=e^{-x_1-x_2-x_3}$ and $f(x)=e^{x_1+x_2+x_3}$,

By Equation (3.24):

$$u_0(x_1,x_2,x_3,\ t)=L^{-1}\left[\frac{1}{S^\beta}\left[u^{\,0}(x_1,x_2,x_3,0)s^{\beta-0-1}\right]\right]=\ L^{-1}\left[\frac{1}{S^\beta}\left[\left(e^{x_1+x_2+x_3}\right)s^{\beta-0-1}\right]\right]=\ L^{-1}\left[\frac{1}{S}\times\left(e^{x_1+x_2+x_3}\ \right)\right]=e^{x_1+x_2+x_3}$$

$$u_0(x_1,x_2,x_3\ \ t)=e^{x_1+x_2+x_3} \qquad (3.48\ )$$

By Equation (3.25):

$$u_1((x_1,x_2,x_3,\ t)=L^{-1}\left[\frac{1}{S^\beta}L\left[\left(1\times\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\right)(e^{x_1+x_2+x_3})\ -\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_3}\right)\left[\left(e^{-x_1-x_2-x_3}\right)\times\left(e^{x_1+x_2+x_3}\right)\right]\right)\right]\right]$$

$$u_1((x_1,x_2,x_2,t)=L^{-1}\left[\frac{1}{S^\beta}L\left[e^{x_1+x_2+x_3}+e^{x_1+x_2+x_3}+e^{x_1+x_2+x_3}-0\right]\right]=L^{-1}\left[\frac{1}{S^\beta}\times\frac{1}{s}\times\left[3e^{x_1+x_2+x_3}\ \right]\right]=\frac{3e^{x_1+x_2+x_3}t^\beta}{\Gamma(\beta+1)}$$

$$u_1(x_1,x_2,x_3,t)=\frac{3e^{x_1+x_2+x_3}t^\beta}{\Gamma(\beta+1)},\quad 0<\beta\le 1, x_1>0, x_2>0, x_3>0, t>0 \qquad (3.49)$$

By Equation (3.26):

For $p=1$, $u_2=\ L^{-1}\left[\frac{1}{S^\beta}L\left[D\frac{\partial^2}{\partial x_1^2}(u_0+u_1-u_0)-\frac{\partial}{\partial x_1}F(x_1)(u_0+u_1-u_0)\ \right]\right]$

$$u_2(x_1,x_2,x_3,t)=L^{-1}\left[\frac{1}{S^\beta}L\left[\left(1\times\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)\left(e^{x_1+x_2+x_3}+\frac{3e^{x_1+x_2+x_3}t^\beta}{\Gamma(\beta+1)}-\left(e^{x_1+x_2+x_3}\right)\right)\ -\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}\right)\left[\left(e^{-x_1-x_2-x_3}\right)\times\left(e^{x_1+x_2+x_3}+\frac{3e^{x_1+x_2+x_3}t^\beta}{\Gamma(\beta+1)}-\left(e^{x_1+x_2+x_3}\right)\right)\right]\right)\right]\right]$$

$$u_2(x_1,x_2,x_3,t)=L^{-1}\left[\frac{1}{S^\beta}L\left[\frac{3^2\left(e^{x_1+x_2+x_3}\right)t^\beta}{\Gamma(\beta+1)}\ \right]\right]=L^{-1}\left[\frac{1}{S^\beta}\times\frac{3^2\times e^{x_1+x_2+x_3}}{s^{\beta+1}}\right]=\frac{3^2\times e^{x_1+x_2+x_3}t^{2\beta}}{\Gamma(2\beta+1)}$$

$$u_2(x_1,x_2,x_3,t)=\frac{3^2\times e^{x_1+x_2+x_3}t^{2\beta}}{\Gamma(2\beta+1)},\quad 0<\beta\le 1, x_1>0, x_2>0, x_3>0,\ t>0 \qquad (3.50)$$

Continuing with this process, we obtain that:

$$u_i(x_1,x_2,x_3,t)=u_{p+1}(x_1,x_2,x_3,t)=\frac{3^i\times e^{x_1+x_2+x_3}t^{i\beta}}{\Gamma(i\beta+1)},\quad 0<\beta\le 1,\ \ x_1>0,\ \ x_2>0,\ \ x_3>0,\ t>0,\ i=1,2,3,\cdots,P+1, P\in IN \qquad (3.51\ )$$

.

Then the i^{th} order approximate solution of Equation $(3.47a)$ given that $(3.47b)$, denoted by $\tilde{u}_i(x_1,x_2,x_3,t)$ is given by:

$$\tilde{u}_i(x_1,x_2,x_3,t)=\sum_{i=0}^{p+1}\frac{3^i e^{x_1+x_2+x_3} t^{i\beta}}{\Gamma(i\beta+1)}, 0<\beta\le 1, x_1>0, \ x_2>0, \ x_3>0, t>0 \qquad (3.52)$$

By letting $p\in IN$ to ∞ or taking limit of both sides of Equation (3.52) as $p\in IN\to\infty$, the closed solution of Equation $(3.47a)$ in the form of infinite fractional power series denoted by $u(x_1,x_2,x_3,t)$ is:

$$u(x_1,x_2,x_3,t)=\sum_{i=0}^{\infty}\frac{3^i e^{x_1+x_2+x_3} t^{i\beta}}{\Gamma(i\beta+1)}, 0<\beta\le 1, x_1>0, \ x_2>0, \ x_3>0, t>0 \qquad (3.53)$$

Thus, by using Equation (3.4) in Equation (3.53), the closed solution of Equation $(3.47a)$ in terms of Mittag-Leffler function of one parameter is given by:

$$u(x_1,x_2,x_3,t)=e^{x_1+x_2+x_3}E_\beta\left(3t^\beta\right), 0<\beta\le 1, x_1>0, \ x_2>0, \ x_3>0, t>0 \qquad (3.54)$$

Lastly, the exact solution of Equation $(3.47a)$, $u_{exact}(x_1,x_2,x_3,t)$ can be obtained from Equation (3.54) as β approaches to 1 from left and it is given by

$$u_{exact}(x_1,x_2,x_3,t)=e^{x_1+x_2+x_3}e^{3t}, \beta=1, x_1>0, \ x_2>0, \ x_3>0, t>0 \qquad (3.55)$$

In order to show the agreement between the exact solution, Equation (3.55) and the i^{th} order approximate solution, Equation (3.52) of Equation $(3.47a)$ given that $(3.47b)$, the absolute errors: $E_4(u)=\left|u_{exact}(x_1,x_2,x_3,t)-\tilde{u}_4(x_1,x_2,x_3,t)\right|$ and $E_5(u)=\left|u_{exact}(x_1,x_2,x_3,t)-\tilde{u}_5(x_1,x_2,x_3,t)\right|$ were computed as shown below by tables3.5 and3.6 by considering the 4^{th} order approximate solutions , $\tilde{u}_4(x_1,x_2,x_3,t)=\sum_{k=0}^{4}\frac{3^i e^{x_1+x_2+x_3} t^{i\beta}}{\Gamma(i\beta+1)}, x_1,x_2,x_3,t,\beta\in\left\{\frac{1}{3},\frac{2}{3},1\right\}$ and the 5^{th} order approximate solutions ,$\tilde{u}_5(x_1,x_2,x_3,t)=\sum_{i=0}^{5}\frac{3^i e^{x_1+x_2+x_3} t^{k\beta}}{\Gamma(k\beta+1)}, x_1,x_2,x_3,t,\beta\in\left\{\frac{1}{3},\frac{2}{3},1\right\}$ of Equation $(3.47a)$ given that $(3.47b)$ without loss of generality.

Table3.5. *Absolute error of approximating the solution of Equation(3.47a) given that (3.47b) to4thorderusing IFLTM*

Variables				Absolute error, $E_4(u)=\left\|u_{exact}(x_1,x_2,x_3,t)-\tilde{u}_4(x_1,x_2,x_3,t)\right\|$		
t	x_1	x_2	x_3	$\alpha=\frac{1}{3}$	$\alpha=\frac{2}{3}$	$\alpha=1$
$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	81.894351	11429758	0.027033
$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	222.611942	30.570542	0.073517

$\frac{1}{3}$	1	1	1	605.121992	84.455123	0.199811
$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	167.919612	23.956946	8.446234
$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	456.45283	65.121748	11.479623
$\frac{2}{3}$	1	1	1	1240.767433	177.01926	20.803233
1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	242.672618	69.275518	10.086288
1	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	659.652584	188.310399	27.417373
1	1	1	1	1793.121606	511.880752	74.528128

Table 3.6.Absolute error of approximating the solution of Equation(3.47a)given that(3.47b) to5thorder using IFLTM

Variables				Absolute error, $E_5(u)=\left\|u_{exact}(x_1,x_2,x_3,t)-\tilde{u}_5(x_1,x_2,x_3,t)\right\|$		
t	x_1	x_2	x_3	$\alpha=\frac{1}{3}$	$\alpha=\frac{2}{3}$	$\alpha=1$
$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	148.959841	13.261491	0.165307
$\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	404.914862	36.048454	0.449334
$\frac{1}{3}$	1	1	1	1100.672701	97.989863	1.221402
$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	397.692495	47.051958	8.446234
$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	1081.040267	127.900466	11.479623
$\frac{2}{3}$	1	1	1	2938.572099	347.669517	20.803233
1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	681.69511	140.604345	4.581767

1	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	1853.039446	382.20222	12.454535
1	1	1	1	5037.083448	1038.933356	33.854916

DISCUSSION

Here, the results obtained from the three application examples considered above are discussed. Through the three examples above, the iterative fractional Laplace transform method (IFLTM)was successfully applied to the time fractional diffusion equations, that is, Equation $(1.2a)$ given that $(1.2b)$, for $F(x_1)=-x_1$ with initial conditions $f(x_1)=1$, $F(x_1,x_2)=-x_1-x_2$ with initial conditions $f(x_1,x_2)=x_1+x_2, F(x_1,x_2,x_3)=e^{-x_1-x_2-x_3}$ with initial conditions $f(x_1,x_2,x_3)=e^{x_1+x_2+x_3}$, $\lambda=1$ and $0<\beta\le 1$.

As a result, through example one, the closed solutions of Equation $(1.2a)$ given that $(1.2b)$ in the form of infinite fractional power series and in terms of Mittag-Leffler function in one parameter as well as its exact solution were obtained and they are in complete agreement with the results obtained byCetinkaya and Kiymaz(2013), kebede(2018), Kumar et al.(2012)and A. Kumar et al.(2017). For $\beta=\frac{1}{2}$ with $F(x_1),\lambda$ and $f(x_1)$ specified in example one above, the closed solutions of equation $(1.2a)$ given that $(1.2b)$ in the form of infinite fractional power series and in terms of Mittag-Leffler function in one parameter as well as the irex act solution, which were obtained by IFLTM, are in complete agreement with the results obtained bykebede(2018)and S. Das (2009).

From the application of IFLTM to Equation $(1.2a)$ given that $(1.2b)$ through the second and third examples above, where $F(x_1,x_2)=-x_1-x_2$ with initial condition $f(x_1,x_2)=x_1+x_2$, $F(x_1,x_2,x_3)=e^{-x_1-x_2-x_3}$ with initial condition $f(x_1,x_2,x_3)=e^{x_1+x_2+x_3}$ and $0<\beta\le 1$,the closed solutions in the form of infinite fractional power series and in terms of Mittag-Leffler function in one parameter as well as exact solution were obtained.

Without loss of generalitythe5thand6thorder approximate solutions of Equation $(3.27a)$; $\forall(x_1,t)\in\{0.25,0.5,0.75,1\}\times\{0.25,0.5,0.75,1\}$, $\forall\beta\in\{0.25,0.5,0.75,1\}$, and the 4rd and5thorder approximate solutions of Equations: $(3.37a)$ and $(3.47a)$

$$\forall(x_1,x_2,t,)\in\left\{\frac{1}{3},\frac{2}{3},1\right\}\times\left\{\frac{1}{3},\frac{2}{3},1\right\}\times\left\{\frac{1}{3},\frac{2}{3},1\right\}$$

and

$$\forall(x_1,x_2,x_3,t,)\in\left\{\frac{1}{3},\frac{2}{3},1\right\}\times\left\{\frac{1}{3},\frac{2}{3},1\right\}\times\left\{\frac{1}{3},\frac{2}{3},1\right\}\times\left\{\frac{1}{3},\frac{2}{3},1\right\}$$

respectively were considered to compute absolute errors in this paper. The validity, accuracy and convergence of the IFLTM was checked through the computed absolute errors:

$$\begin{cases}E_5(u)=\left|u_{exact}(x_1,t)-\tilde{u}_5(x_1,t)\right|\\ E_6(u)=\left|u_{exact}(x_1,t)-\tilde{u}_6(x_2,t)\right|\end{cases};$$
$$\forall\beta\in\{0.25,0.5,0.75,1\}\subseteq(0,1]$$

$$\begin{cases}E_4(u)=\left|u_{exact}(x_1,x_2,t)-\tilde{u}_4(x_1,x_2,t)\right|\\ E_5(u)=\left|u_{exact}(x_1,x_2,t)-\tilde{u}_5(x_1,x_2,t)\right|\end{cases};$$
$$\forall\beta\in\left\{\frac{1}{3},\frac{2}{3},1\right\}\subseteq(0,1],$$

$$\begin{cases}E_4(u)=\left|u_{exact}(x_1,x_2,x_3,t)-\tilde{u}_4(x_1,x_2,x_3,t)\right|\\ E_5(u)=\left|u_{exact}(x_1,x_2,x_3,t)-\tilde{u}_5(x_1,x_2,x_3,t)\right|\end{cases}$$
$$;\forall\beta\in\left\{\frac{1}{3},\frac{2}{3},1\right\}\subseteq(0,1]$$

,where $u_5(x_1,t)$ is the 5th order approximate solutions, $u_6(x_1,t)$ is the 6th order approximate solutions and $u_{exact}(x_1,t)$ is the exact solutions of example one; $u_4(x_1,x_2,t)$ is the 4th order approximate solutions, $u_5(x_1,x_2,t)$ is the 5th order approximate solutions and $u_{exact}(x_1,x_2,t)$ is the exact solution of example two; $u_4(x_1,x_2,x_3,t)$ is the 4th order approximate solutions, $u_5(x_1,x_2,x_3,t)$ is the 5th order approximate solutions and $u_{exact}(x_1,x_2,x_3,t)$ is the exact solution of example three. From observation made through tables 3.1 to 3.6,the absolute errors: $E_5(u)$ and $E_6(u)$ decrease as $\beta \in \{0.25, 0.5, 0.75, 1\}$ increases from 0.25 to 1; $E_4(u)$ and $E_5(u)$ decrease as $\beta \in \left\{\frac{1}{3}, \frac{2}{3}, 1\right\}$ increases from $\frac{1}{3}$ to 1. These imply that the 5th order approximate solutions and the 6th order approximate solutions of Equation (3.27a)converge to their exact solution as $\beta \in \{0.25, 0.5, 0.75, 1\}$ increases from 0.25 to 1; the 4th order approximate solutions and the 5th order approximate solutions of Equations(3.37a)and (3.47a)converge to their exact solutions as $\beta \in \left\{\frac{1}{3}, \frac{2}{3}, 1\right\}$ increases from $\frac{1}{3}$ to 1.It was also observed that $E_5(u) > E_6(u)$ for each $(x_1,t) \in \{0.25, 0.5, 0.75, 1\} \times \{0.25, 0.5, 0.75, 1\}$ and for each $\beta \in \{0.25, 0.5, 0.75, 1\}$ throughout tables: 3.1 and 3.2; $E_4(u) > E_5(u)$ for each $(x_1,x_2,t,) \in \left\{\frac{1}{3}, \frac{2}{3}, 1\right\} \times \left\{\frac{1}{3}, \frac{2}{3}, 1\right\} \times \left\{\frac{1}{3}, \frac{2}{3}, 1\right\}$ and for each $\beta \in \left\{\frac{1}{3}, \frac{2}{3}, 1\right\}$ throughout tables: 3.3 and 3.4; $E_4(u) > E_5(u)$ for each $(x_1,x_2,x_3,t,) \in \left\{\frac{1}{3}, \frac{2}{3}, 1\right\} \times \left\{\frac{1}{3}, \frac{2}{3}, 1\right\} \times \left\{\frac{1}{3}, \frac{2}{3}, 1\right\} \times \left\{\frac{1}{3}, \frac{2}{3}, 1\right\}$ and for each $\beta \in \left\{\frac{1}{3}, \frac{2}{3}, 1\right\}$ throughout tables: 3.5 and 3.6.

These show that the validity, accuracy and convergence of the fractional power series solutions of equations (3.27a), (3.37a) and (3.47a) can be improved by calculating more term in the series solutions by using the present method, IFLTM.

CONCLUSION

In this study, basic idea of iterative fractional Laplace transform method (IFLTM) for solving(n+1) dimensional time fractional diffusion equations with initial conditions of the form(1.2a) given that (1.2b) was developed. The IFLTM was applied to three (n+1) dimensional time fractional diffusion equations with initial conditions to obtain their closed solutions in the form of infinite fractional power series andin terms of Mittag-Leffler functions in one parameter which rapidly converge to exact solutions. The closed solutions in the form of infinite fractional power series and in terms of Mittag-Leffler functions in one parameter, which rapidly converge to exact solutions, were successfully derived by the use of iterative fractional Laplace transform method (IFLTM). The results evaluated for the first time fractional diffusion equations is in a good agreement with the one already existing in the literature. Precisely, IFLTM works successfully in solving time fractional diffusion equations with initial conditions to obtain their closed solutions in the form of infinite fractional power series andin terms of Mittag-Leffler functionin on parameter as well as exact solutions with a minimum size of calculations.

Thus, we can conclude that the IFLTM used in solving time fractional diffusion equations with initial conditions can be extended to solve other fractional partial differential equations with initial conditions which can arise in fields of sciences.

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