International Journal of Research Studies in Science, Engineering and Technology
Volume 5, Issue 3, 2018, PP 7-10
ISSN : 2349-476X

Existence and Uniqueness of Solutions of Delay Systems with Non-Critical Linear Part

Nwaoburu A.O

Department of Mathematics, Faculty of Science, Rivers State University of Science and Technology, P.M.B. 5080, Nkpolu- Oroworukwo, Port Harcourt, Rivers State, Nigeria.

Corresponding Author: Nwaoburu A.O, *Department of Mathematics, Faculty of Science, Rivers State University of Science and Technology, P.M.B. 5080, Nkpolu- Oroworukwo, Port Harcourt, Rivers State, Nigeria.*

ABSTRACT

To show that the system in question can be non-critical and hence provide the Green Matrix solution for such a system and therefore proceed to finding solution of the perturbation of the system.

INTRODUCTION

Conditions for the existence and uniqueness of solutions for differential equations are of immense importance in the fields of Mathematics, Science and Engineering and constitute the core contents. Researches on the subject have been intensive. Literature is vast on the existence and uniqueness of solutions of the ordinary differential equations with initial conditions given by

$$y = f(x, y);\ y(x_0) = y_0 \tag{1}$$

Peano, Picard and other mathematical geniuses have provided a set of conditions for the existence and uniqueness of solutions of system (1.1). These conditions include;

(i) the continuity on *f* and

(ii) the satisfaction of Lipschitz condition by *f*.

Interest on the subject has diversified to accommodate investigations on the existence and uniqueness of systems with boundary values and systems with periodic solutions.

Quite recently, results on existence and uniqueness of solutions have been carried over to functional differential systems. Chukwu [4], Cruz and Hale [5], Driver [7], On wuatu [9], Hale [6] have investigated necessary and sufficient conditions for the existence and uniqueness of solutions of delay and neutral systems. Hale [6] provided sufficient conditions for the global existence and the exponential estimates of solutions of non-linear delay systems of the form

$$\begin{aligned} \dot{x}(t) &= L(t, x_t) + b(t); && t > \delta \\ xt &= \phi; && t\ \varepsilon\ [\delta,\ \infty] \end{aligned} \tag{2}$$

The generalised neutral functional inclusion

$$\left(\frac{d}{dt}\right) D(t, x_t)\ \varepsilon\ R(t, x_t) \tag{3}$$

Was studied by Chukwu [4]. He under certain regularity conditions on D and R and invoking K. Fan's fixed point theorem, provided the existence of solution of equation (1.3) which satisfies two point boundary values $x_{t_0} = \phi_0$ *and* $x_{t_1} = \phi_1$;

Most recently, Ukwu [1] inspired by the studies of Roseau [2] and Rouche & Mawhin [3] has provided conditions for the existence of the solution of the parameter dependent system

$$\dot{x} = A(t)x + B(t)u + f(t, x, u);\ t\ \varepsilon\ [t_0, t_0 + T] \tag{4}$$

with boundary condition $x(t_0) = x(t_0 + T_0)$.

He obtained a variation of constant formula for the system, and utilising the condition of existence of a T-periodic solution for the homogeneous part of (1.4), he defined the Green matrix whose properties were exploited to establish a solution of the linear perturbation (1.4) satisfying the boundary values.

Systems with periodic solutions are of practical relevance in the study of boundary value problems, theory of oscillation, and invariably motivate interest and attention in Physics and Engineering.

An investigation into an analogous system of (1.4) of the form

$$\dot{x}(t) = L(t, x_t) + f(t, x_t) \qquad t \; \varepsilon \; [0, 1]$$

$$x_0 = x(T), \; T > 0 \qquad (5)$$

is therefore expedient, and is the main objective of our research.

Notations and Preliminaries

Let E denote the real line. For a positive integer n, E^n denotes the space of real n – tuples with the usual Euclidean norm $I \bullet I$ and $C([a, b], E^n)$ is the Banach space of continuous functions from $[a, b]$ into E^n with the supremum norm. The norm of ϕ in $C([-h, 0], E^n)$ is given by

$$\|\phi\| = \sup_{-h \le s \le 0} |\phi(s)|$$

In this paper the state space will be $C([-h, 0], E^n)$ with norm defined by

$$\|x\| = \max_{0 \le t \le T} |x(t)| \quad with \; h > 0, if \; x : [-h, t] \to E^n \; and$$

$t \; \varepsilon \; [0, T]$, the symbol x_t denotes the function on $[-h, 0]$

With $x_t(s) = x(t+s); \; s \, \varepsilon [-h, 0], \; h > 0$

Consider the system

$$\dot{x}(t) = L(t, x_t) + f(t, x_t) \qquad -h \le t \le \infty \qquad (6)$$

With

$x(0) = x(T), \; where \; x(t) = \phi(t), \; \phi \, \varepsilon \, C \, t \, \varepsilon [-h, 0]; h > 0$

$$L(t, \phi) = \sum_{k=0}^{\infty} A_k(t)\, \phi(-t_k) + \int_{-h}^{0} A(t, s)\phi(s)\, ds$$

$L(t, \phi)$ is assumed continuous in t, linear in ϕ and in each A_k, $A(t, s)$ is a continuous matrix function for $-\infty < t, \; s < \infty, \; 0 < t_k, \; \tau \le h.$

The function $f : J \times C \to E^n$ is continuous.

Denote by $X(t)$, the principal fundamental matrix of the linear part of (2.1)

$$\dot{x} = L(t, x_t) \qquad (7)$$

By the variation of constant formula with $x(0) = \phi(0) = \alpha, a \, \varepsilon \, E^n$ the solution of (2.1) can be expressed as

$$x(t) = X(t)\, X^{-1}(0)\, \phi(0) + X(t) \int_0^t X^{-1}(\tau) f(\tau, s) dy$$

$$x(t) = X(t)[\alpha + \int_0^t X^{-1}(\tau)\, f(\tau, s) d\tau \qquad (8)$$

$$x(T) = X(T)[\alpha + \int_0^t X^{-1}(\tau)\, f(\tau, s) d\tau]$$

Since $x(0) = x(T) = \alpha$, *we have*

$$\alpha = X(T)[\alpha + \int_0^T X^{-1}(\tau)\, f(\tau, s) d\tau$$

$$|I - X(T)|\alpha = X(T) \int_0^T X^{-1}(\tau)\, f(\tau, s)\, dt$$

Clearly, if $[I - X(T)$ is invertible, we have

$$\alpha = [T - X(T)]^{-1} \int_0^T X(T) X^{-1}(\tau)\, f(\tau, s) d\tau$$

Substitution this value of α into (2.3) gives the corresponding T periodic solution of (2.2) we shall next show that the non-singularity of $[I - X(T)]$ shows that

$$\dot{x}(t) = L(t, x_t), \qquad x(0) = x(T)$$

Admits only the trivial solution.

Proposition

Consider the system

$$\dot{x}(t) = L(t, x_t), \qquad x(0) = x(T)$$

the system admits only the trivial solution, if and only if the determinant of $[I - X(T)] \neq 0$.

Proof

The solution of (2.2) has the form

$$x(t) = X(t)\, X^{-1}(0)\, \phi(0) = X(t)\, \alpha$$

$$X(T) = X(T)\alpha \qquad \Rightarrow \qquad \alpha = X(T)\alpha.$$

Clearly, $[I - X(T)]\alpha = 0$

Since $[I - X(T)] \neq 0, \alpha = 0$

The system admits only trivial solution. The converse is easily seen. When this is the case, the linear part of system (2.1) is said to be NON-CRITICAL.

The Green Matrix

Substituting the value of α into (2.3) gives

$$x(t) = X(t)[I - X(T)]^{-1} \int_0^1 X(T) X^{-1}(\tau)\, f(\tau, s) d\tau \; +$$

$$\int_0^t X(t) X^{-1}(\tau) \; f(\tau, s) d\tau \; + \; X(t)[1 - X(T)] \int_t^T X(T) X^{-1}(\tau)\, f(\tau, s) d\tau$$

$$= \int_0^T G(\tau, s) \; f(\tau, s) d\tau$$

Where $G(t, s)$ is an $n \times n$ matrix function called the green matrix of the system and is defined by

$$G(t,s)=\begin{cases} X(t)[I-X(T)]^{-1} X(T)X^{-1}(s) + X(t)X^{-1}(s) \\ \qquad if\ 0 \le s \le t \le T \\ X(t)[1-X(T)]^{-1} X(T)X^{-1}(s) \\ \qquad if\ t \le s \le T \end{cases}$$

Simplifying further, we have

$$G(t, s) = \begin{cases} X(t)[I-X(T)]^{-1} X^{-1}(s) \\ \qquad if\ 0 < s < t < T \\ X(t)[I-X(T)]^{-1} X(T) X^{-1}(s) \\ \qquad if\ t < s \le T \end{cases}$$

Evidently, we have the following properties of the green matrix.

- $G(t, s)$ is differentiable almost everywhere for $t\ \varepsilon [0,T]-\{s\}$ and $\frac{dG}{dt} = A(t)G$
- G(t, s) is continuous except at t = s $G(s+0,s)-G(s-0,s)=I$ (*Identity marix*)

Hence, G is bound and continuous almost everywhere.

$$G(0,s)=G(T,s)$$

Definition

The solution $x(t)$ of 2.1 defined on E such that

$$x(t+T)=x(t) \tag{9}$$

for all $t\ \varepsilon\ E$ is called a T – periodic solution

Banach Fixed Point Theorem

Let $T: C\gamma \to C\gamma$ be a mapping from a bounded set to another bounded set. If T is a contraction mapping, it has a fixed point in $C\gamma$, that is for some $x\,\varepsilon\, C\gamma$

$$Tx = x$$

EXISTENCE AND UNIQUENESS OF SOLUTIONS OF PERTURBATIONS OF DELAY SYSTEMS WITH NON-CRITICAL LINEAR PART

Consider the delay system

$$\dot{x}(t)=L(t,x_t)+f(t,x_t) \tag{10}$$

with $x(0)=x(T)$

We assume that the homogenous equation

$$\dot{x}(t)=L(t,x_t)+x(0) \quad =x(T) \tag{11}$$

is non-critical. This implies the existence of the Green matrix $G(t, s)$ for the systems. The integral equation, the equivalent of the equation can thus be written as

$$\dot{x}(t)=\int G(t,s)\ f(s,x)ds$$

We shall present sufficient conditions which guarantee the existence and uniqueness of the solution of the given equation which satisfies the boundary values.

Existence and Uniqueness Theorem

In this section, we give sufficient conditions on f to ensure the existence of the solution of (3.1)

Theorem

Consider the delay system

$$\dot{x}(t)=L(t,x_t)+f(t,x_t)$$

with boundary $x(0)= x(T)$

Suppose

- The linear part is not critical
- f is Lipschitzian with respect to x for all $t\ \varepsilon\ [0, T]$ that exist $\lambda > 0$, $R > 0$ such that

$$|f(t,x_t)-f(t,y_t)| \le \lambda |x_t - y_t|; |x_t| \le R, |y_t| \le R$$

Also there exist $\gamma > 1$ and $\delta < (1 - \gamma)R$ where

$$\gamma = \lambda \sup_{t\,\varepsilon|0,T|} \int_0 |G(t,s)|ds; \quad \gamma < \infty$$

and $$d=\sup_{t\,e|0,T} \int_0^T |G(t,s)||f(s,0|ds$$

then the boundary value problem has a unique T – periodic solution.

Proof

Let C be the Banach space of all continuous functions $x:[-h,T] \to E^n$ such that $x(0)=x(T)$

Let C_R be a compact ball in C of radius R. For sufficiently large R, C_R can be chosen non-empty.

We now defined an operator.

$L: C_R \to C_R$ Given as

$$L[x(t)]=\int_0^T G(t,s)\ f(s,x_s)ds$$

And then prove that L is a contraction.

We first prove that L is well defined, that is for $x \varepsilon C_R$, $L(x) \varepsilon C_R$. Let $x \varepsilon C_R$ then

$$|L(x)(t)| = \left|\int_0^T G(t,s) f(s,x_s)ds\right| = \left|\int_0^T G(t,s)[f(s,x_s) - f(s,x_s) + f(s,0)]ds\right|$$

$$\leq \int_0^T |G(t,s)||f(s,x_s) - (f(s,0)| + \int_0^T |G(t,s)||f(s,x_s)|ds$$

$$\leq \lambda \sup \int_0^T |G(t,s)|ds\, \|x\|c + \delta$$

$$t\,\varepsilon|0,T|$$

$$= \gamma\|x\|c + \delta \leq R\,(from(iii)).$$

Clearly, $\|L(x)\| < R,$ showing that $L(x)(t)\,\varepsilon\, C_R$

Hence, L is well defined.

Next, we show that L is a contraction on C_R.

$$|L(x)(t) - L(y)(t)| < \int_0^T |G(t,s)||f(s,x_s) - f(s,y_s)|ds$$

$$\leq \lambda \int_0^T |G(t,s)||x_s - y_s|ds\,(by\,(ii))$$

The Supremum both sides over *[0, T]* to have

$$L[x_t] - L[y_t] \leq (\lambda \sup \int_0^T |G(t,s)ds\|x_s - y_s|$$

$$< \gamma\|x_s - y_s\|\ for\ x_s, y_s\ \varepsilon\, C_R$$

Since $\gamma < 1$, L is shown to be a contraction.

Having met the conditions for the application of Banach's fixed point theorem, we conclude that has a fixed point. That is, $L(x)\ (t) = x(t)$. This fixed point is the required solution for the boundary value problem.

Conclusion

The proofs of Existence and Uniqueness theorems are usually lengthy. However, utilising the properties of the ***green matrix***, a finite set of conditions has been provided for a precise proof of the existence and uniqueness of solutions of periodic boundary value problems under reference, and this is one of the merits of this research.

References

[1] Chukwu E. N. Functional Inclusion and Controllability of Non-linear Neutral Functional Differential Systems. J. O. T. A.

[2] Driver R. D. A Functional Differential System of Neutral System of Neutral Type in a Two Body Problem of Classical Electrodynamic in Non-linear Differential Equations and Non linear Mechanics – Academics Press, New York, 1963.

[3] Hale J. K. Theory of Differential Equation, Springer –Verlag, New York, 1977

[4] Hale J. K. and Cruz M. A. Existence, Uniqueness and Continuous Dependence for Hereditary Systems. Annali di Mathematica pura ed Applicata, Vol. 85, pp 63 – 82, 1970.

[5] M. Roseau Equations Differentielels Masson, 1976.

[6] R. Rouche, J. Mawhin "Equation Differentials Ordinaires" Masson at Editeur 120 Boulevardm Saint German, Paris, Vol 1, 1973.

[7] Onwuatu J. U. Null Controllability of Non-linear Infinite Neutral System, KYBERNETIKA Vol. 29 no x 1993.

[8] Stephen G. Retarded Dynamical Systems Stability and Characteristic function John/Wilan, New York.

[9] Ukwu C. Complete Controllability of Periodic Boundary value Problem with Non-critical Free Part, 1986. J. O. T. A.

Citation: *A. Nwaoburu, "Existence and Uniqueness of Solutions of Delay Systems with Non-Critical Linear Part", International Journal of Research Studies in Science, Engineering and Technology, vol. 5, no. 3, pp. 7-10, 2018.*

Copyright: *© 2018 A. Nwaoburu, et al. This is an open-access article distributed under the terms of the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original author and source are credited.*