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Solutions to Landau's Problems and Other Conjectures on Prime Numbers

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Abstract: *In Number Theory there are many conjectures related to prime numbers without demonstration, mainly because the order in which prime numbers form was unknown until Porras-Ferreira and Andrade (2014) revealed it with the solution to the Goldbach's Conjectures.*

This manuscript presents solutions to three of the four problems or conjectures expressed by Landau during the Fifth International Congress of Mathematics in 1912 as "Unattackable at the actual state of science": The Conjecture of the Twin Prime Numbers, Legendre's Conjecture and the Conjecture on the existence of infinite prime numbers p, such as p-1, is a perfect square. Furthermore, the solutions to Conjecture 1379 and Brocard's Conjecture are presented. The exact accomplishment of each one of the conjectures confirms the Prime Numbers Order found in Porras-Ferreira and Andrade (2014).

Keywords: *Landau´s Problems, Twin Primes Conjecture, Conjecture 1379, Legendre's Conjecture, infinite Prime Numbers of the forma* $^2 + 1$, *Brocard's Conjecture, Prime Numbers.*

1. Introduction

In the International Congress of Mathematics celebrated in Cambridge in 1912, in (Curbera, 2007) [1], Edmund Landau listed four problems related to Prime Numbers that he stated were "unattackable at the present state of science." The problems, which end up designated as the "Landau´s Problems" are:

1. The Twin Primes Conjecture "*¿Exist and infinite number of Primes p such that p+2 is also a prime?*" (e.g, Hardy and Littlewood, 1929) [2].
2. The Goldbach's Conjecture:"*¿Every even number greater than 2 can be written as the sum of two Prime Numbers?*" (Goldbach, 1742) [3].
3. The Legendre's Conjecture: "*¿For all natural number n there is at least one Prime Number between* n^2 *and*$(n+1)^2$?"(*e.g.* Chen, 1975 [4]; Hardy and Wright, *1979* [5]).
4. *"¿Are there infinite Prime Numbers of the forma*$^2 + 1$?" (e.g. Euler, 1760) [6].

Likewise, there are other conjectures on the Prime Numbers without solutions such as:

- The Conjecture 1379: "*¿Are there infinite Primes ending in 1, 3, 7 and 9 and continuous?"* (Porras-Ferreira, 2012) [7].
- The Brocard's Conjecture: "*¿Are there at least four Prime Numbers in between* $(p_n)^2$ *and* $(p_{n+1})^2$, *for n>1, where* p_n *is the n-emsim Prime Number*?" (Wells, 2005) [8].

Specialized literature has been filled with manuscripts showing many efforts to find solutions to these problems, but they remain unsolved in the mathematics of numerical analysis.

Recent achievements studying these conjectures on primes include the exploration of its short intervals (e.g Pintz, 1981 [9], 1984 [10]; Watt, 1995 [11]) in large intervals (Pintz, 1997) [12], the difference between consecutive primes (Baker et al., 2001) [13] and on small gaps between them (Goldston et al., 2006) [14]. Also, the studies on exceptional sets of twin primes, (e.g. Perelli, and Pintz, (1992) [15] together with the more recent finding of a finite limit for the gap between twin primes in Zhang (2014) [16], brought a different vision on the rhythm for prime numbers to appear.

Looking after a simpler pattern, an order for prime numbers was found and presented in Porras-Ferreira and Andrade (2014) [17]. Furthermore, two independent solutions to the Goldbach's

Conjecture (problem 2) extended to the "weak Conjecture" were proposed in [17] using simple algebraic statements based on the regularities found in the formation of Primes. Also expressed as:

$$[31, 7, 11, 13, 17, 19, 23, 29] + 30n, \text{ for } n \geq 0 \quad (1)$$

The exceptions are the numbers 2, 3 and 5, which are the only primes not found in the pattern.

Furthermore, a similar expression to the equation (1) is identical using Modular Identities; meaning that all primes p exept 2, 3, 5, 7, 11, 13, 17, 19, 23, 29 and 31, only have the following modular identities:

$$p \equiv \begin{bmatrix} 7\ (mod\ 30) \\ 11\ (mod\ 30) \\ 13\ (mod\ 30) \\ 17\ (mod\ 30) \\ 19\ (mod\ 30) \\ 23\ (mod\ 30) \\ 29\ (mod\ 30) \\ 31\ (mod\ 30) \end{bmatrix} \quad (2)$$

Since primes are infinite, each modular identity contains infinite primes also. See Figure 1.

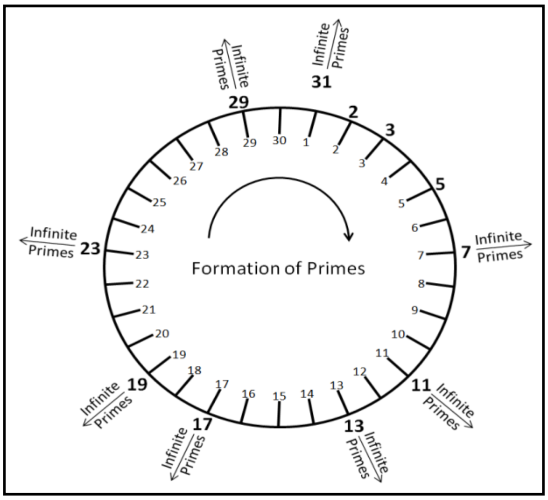

Fig1. *Prime formation clock (mod 30), according with Equation 2.*

Modular arithmetic is referenced in number theory, group theory, ring theory, abstract algebra, knot theory, cryptography, computer algebra, computer science, visual arts and musical arts. In particular, it can be used to obtain information about the solutions, or lack thereof, of a specific equation.

Modular arithmetic can be worked mathematically by introducing a congruence relation on the integers that is compatible with the operations of the ring of integers: addition, subtraction and multiplication. For a positive integer *n*, two integers *a* and *b* are said to be congruent modulo n, written:

$$a \equiv b \ \ (mod\ n)$$

The properties that make this relation a congruence relation, respecting addition, subtraction, and multiplication, are the following.

If

$$a_1 \equiv b_1\ (mod\ n)$$

and

$$a_2 \equiv b_2\ (mod\ n)$$

then:

$$a_1 + a_2 \equiv b_1 + b_2\ (mod\ n)$$

$$a_1 - a_2 \equiv b_1 - b_2 \ (mod\ n)$$

$$a_1 a_2 \equiv b_1 b_2 \ (mod\ n)$$

It should be noted that the addition and subtraction properties would still hold if the theory were expanded to include all real numbers, that is if a_1, a_2, b_1, b_2 and n were not necessarily all integers. However, multiplication would fail if these variables were not all integers:

These basic notions were used along this manuscript to solve the three lasting conjectures of the Landau´s problems. Also, the conjecture 1379[7] is solved using the same found order in [17] and the solution of Brocard's Conjecture is elevated to the category of theorem.

2. Solutions to Landau´s Problems

2.1. Solution to the Twin Primes Conjecture

In [16], Zhang attacked the problem by proving that the number of primes that are less than 70 million units apart are infinite ($\lim_{m\to\infty} inf\ (p_{m+1} - p_m) < 7 * 10^7$). While 70 million is a long way away from 2, Zhang's work marked the first time anyone was able to assign any specific proven number to the gaps between primes. Recently, Polymath8 was launched (Tao, 2014), as a forum where mathematicians could work to reduce that gap between 70 million and 2.They accomplished it to 4,680 within a few months of Zhang submitting his paper [18]. In November 2013, Maynard [19] presented independent work that built on Zhang's to further shrink the gap to 600. The second phase of Polymath8, called Polymath8b, builds on Maynard's work. Currently, the best bound on gaps between primes is 270, and it is believed the work can get down to 6, assuming the generalized Ellioth-Halberstam conjecture [18].

With relation to the conjecture if the Twin Primes are infinite, that is to say, "Exist an infinite number of Primes p such that $p+2$ is also a Prime," corresponding to one of the Landau´s problems, it can be demonstrated as follows:

According to [17] the Prime Numbers order themselves as it is shown in Table 1:

Table1. *The first twenty rows showing the formation (order) the Prime Numbers as they appear in a table of 30 columns*

Rows n	Column 1	Colum 7	Column 11	Column 13	Column 17	Column 19	Column 23	Column 29
0		7	11	13	17	19	23	29
1	31	37	41	43	47		53	59
2	61	67	71	73		79	83	89
3		97	101	103	107	109	113	
4		127	131		137	139		149
5	151	157		163	167		173	179
6	181		191	193	197	199		
7	211			223	227	229	233	239
8	241		251		257		263	269
9	271	277	281	283			293	
10		307	311	313	317			
11	331	337			347	349	353	359
12		367		373		379	383	389
13		397	401			409		419
14	421		431	433		439	443	449
15		457	461	463	467			479
16		487	491			499	503	509
17			521	523				
18	541	547			557		563	569
19	571	577			587		593	599
20	601	607		613	617	619		

The Residue System establishes that any number N divided by another number N ($R = Residue(\frac{N}{n})$ has a residue $R = [0, 1, 2, \ldots, n-1]$. The set of integers $[0, 1, 2, \ldots, n-1]$ is called the least residue system modulo n or modular arithmetic.

Applying the Residue System to Table 1:

- Cells in black indicate the composite numbers multiple of the prime number 7 (residue 0).
- Cells in white indicate the composite numbers that are not multiple of 7 but it is multiple of any other prime number (residue 0).
- Columns 1 and 29, are occupied by composite numbers multiples of 7 in the same rows, because $29 - 1 = 28$ and 28 is multiple of 7. This situation is repeated in the rows $n = 3 + 7k$ for$k \geq 0$.
- There are progressive steps between the cells where there are composite numbers with a multiple of 7 to the following cells, given by the difference between columns divided by 2. For example, in columns 7 and 1 there are $\frac{7-1}{2} = 3$ cells or "steps" of difference where there are composite numbers with multiples of 7 (cell 10 column 1 and cell 7 column 7. In columns 11 and 7 it is obtained $\frac{11-7}{2} = 2$ cells or steps of difference from where there are composite numbers multiples of 7).
- In rows:

$$\begin{bmatrix} n = [0.1, 2, 3, 6] + 7k \text{ for } k \geq 0 \text{ for Columns 11 and 13} \\ n = [0, 3, 4, 5, 6] + 7k \text{ for } k \geq 0 \text{ for Columns 17 and 19} \end{bmatrix} \quad (3)$$

twin primes can exist.

- In rows:

$$n = [0, 3, 6] + 7k \text{ for } k \geq 0 \quad (4)$$

Twin primes can exist at the same row n in columns [11, 13] and [17, 19].

- In addition to the above, for columns 1 and 29, twin primes can exist in the unions of the rows n as follow:

$$\begin{bmatrix} n = 0 + 7k \text{ (column 29) with } n = 1 + 7k \text{ (column 1)} \\ n = 1 + 7k \text{ (column 29) with } n = 2 + 7k \text{ (column 1)} \\ n = 4 + 7k \text{ (column 29) with } n = 5 + 7k \text{ (column 1)} \\ n = 5 + 7k \text{ (column 29) with } n = 6 + 7k \text{ (column 1)} \end{bmatrix} \text{ for } k \geq 0 \quad (5)$$

- The only columns where twin primes do not form are columns 7 and 23.

In a similar way it is possible to apply the Residue System for the next prime numbers in each column. It is not possible to have a composite number with the same prime numbers in the same row and different columns , (except columns 1 and 29 for prime number 7), because the Residue System for those prime numbers will have different residues in the other columns and same row.

If p_m is the m-th prime for columns [11, 17 and 29] and p_{m+1} for columns [13, 19, and 1], according with congruence relation (respecting subtraction):

$$p_{m+1} - p_m \equiv 2 \ (mod\ 30)$$

Since Equations (3) and (5), replicate every 7 rows from $k = 0$ to infinity, there will always be seven rows where twin primes can exist in this replication and the prime numbers which are infinite along every column confirms that there will always be primes in the 8 column array,. It can also be concluded that the twin primes are also infinite, since columns [11, 13] and [17, 19] can have simultaneous prime numbers in the same n row. Also, Equation (3) and columns [29, 1] can have simultaneous prime numbers according to Equation (5). Then:

$$\lim_{m \to \infty} inf\ (p_{m+1} - p_m) = 2$$

The largest known twin prime, discovered in December 2011 is[1]:

$$3756801695685 * 2^{666669} \pm 1$$

[1] The list of the 20 largest known twin primes, can be seen in: http://primes.utm.edu/top20/page.php?id=1

Each one contains 200700 digits. These two primes are generated in columns 29 and 1 taking into account that the number $3756801695685 * 2^{666669}$ end in zero[2], therefore these two primes end in 1 and 29 respectively. See Table 2.

Table2. *Largest known twin prime*

Twin prime form	*n*	Twin prime
30*n*+1 30(*n*-1)+29	250453446379* 2^{666668}	3756801695685* $2^{666669} \pm 1$

Additionally, using the congruence relation (respecting subtraction) with Equation (2) can set that there are:

$$p_{m+1} - p_m \equiv [2, 4, 6, \dots, < \infty] \ (mod\ 30)$$

Then exist:

$$\lim_{m \to \infty} inf\ (p_{m+1} - p_m) = [2, 4, 6, \dots, < \infty]$$

2.1. Solution to Legendre's Conjecture

Legendre's Conjecture: *For all natural numbers n there is at least one prime number between* n^2 *and* $(n+1)^2$, can be solved as follows:

Expressed in mathematical form:

$\boldsymbol{\Pi_l}(n) \geq 1$ between $\boldsymbol{n^2}$ and $(\boldsymbol{n}+\mathbf{1})^{\mathbf{2}}$ where $\boldsymbol{\Pi_l}(n)$ is the amount of prime numbers contained in between these two squared numbers ($\boldsymbol{n^2}$ **and** $(\boldsymbol{n}+\mathbf{1})^{\mathbf{2}}$), it is the same to say:

$\Pi_l(n) = \Pi((n+1)^2) - \Pi(n^2) \geq 1$

Where $\boldsymbol{\Pi}((\boldsymbol{n}+\mathbf{1})^{\mathbf{2}})$ and $\boldsymbol{\Pi}(\boldsymbol{n^2})$ represent the amount of prime numbers between $(\boldsymbol{n}+\mathbf{1})^{\mathbf{2}}$ and $\boldsymbol{n^2}$ respectively.

- The Prime Number Theory established that the amount of prime numbers less than $\boldsymbol{x}$ for very large $\boldsymbol{x}$ is:

$p(x) \cong \frac{x}{ln\ x}$

- Therefore:

$\Pi(n^2) \cong \frac{n^2}{ln\ n^2}$ and $\Pi((n+1)^2) \cong \frac{(n+1)^2}{ln\ (n+1)^2}$

$\Pi_l(n) = \Pi((n+1)^2 - \Pi(n^2) \cong \frac{(n+1)^2}{ln\ (n+1)^2} - \frac{n^2}{ln\ n^2}$

$\Pi_l(n) \cong \frac{n^2}{ln\,(n+1)^2} - \frac{n^2}{ln\ n^2} + \frac{2n+1}{ln\ (n+1)^2}$

- Dividing both terms by $\frac{2n+1}{\ln(n+1)^2}$:

$$\left(\frac{\Pi_l(n)}{\left(\frac{2n+1}{\ln(n+1)^2}\right)}\right) \cong \left(\frac{\frac{n^2}{\ln(n+1)^2} - \frac{n^2}{\ln n^2}}{\left(\frac{2n+1}{\ln(n+1)^2}\right)}\right) + 1$$

- Applying limits to both functions there can be established that:

$$\lim_{n\to\infty}\left(\frac{\Pi_l(n)}{\left(\frac{2n+1}{\ln(n+1)^2}\right)}\right) = \lim_{n\to\infty}\left(\frac{\frac{n^2}{\ln(n+1)^2} - \frac{n^2}{\ln n^2}}{\left(\frac{2n+1}{\ln(n+1)^2}\right)}\right) + 1 = 0 + 1 = 1$$

[2] Because 2 to any power always will end in [2, 4, 6 or 8] and multiplied by 5 (the last number of the mantissa), the result number will end in zero

Note: *the numerator (first term) of the right-hand side of the equation tends to grow very slowly, while the denominator tends to infinite more quickly.*

And

$$\lim_{n\to\infty} \Pi_l(n) = \lim_{n\to\infty}\left(\frac{2n+1}{ln\ (n+1)^2}\right) = \infty$$

- $\Pi_l(n)$ Is also an ascending function, continuous and divergent since it has no limits.
- This can also mean that:

$$\Pi_l(n) \cong \frac{n+1/2}{ln\ \ (n+1)}$$

- If it is possible to show that $\boldsymbol{\Pi_l(n) \geq 1}$ for $\boldsymbol{n \geq 1}$ then Legendre's conjecture would be demonstrated.
- Verifying the above for *n=1*:

$$\Pi_l \cong \frac{n+\frac{1}{2}}{ln\ (n+1)} = \frac{1{,}5}{ln\ 1{,}5} = 2{,}16 > 1$$

Q.E.D[3].

Table 3 verifies the above function with respect to real calculations made in regards to the amount of some prime numbers between $\boldsymbol{n^2}$ and $(\boldsymbol{n+1})^2$.

Table3. *Verifying the function* ***Π_l(n) calculated*** *vs.* ***Π_l(n) real***.

n	n^2	$(n+1)^2$	$\Pi(n^2)$	$\Pi((n+1)^2)$	$\Pi_l$real	Π_l calculated
1	1	4	0	2	2	**2**
2	4	9	2	4	2	**2**
3	9	16	4	6	2	3
4	16	25	6	9	3	3
5	25	36	9	11	2	3
6	36	49	11	15	4	3
7	49	64	15	18	3	4
8	64	81	18	22	4	4
9	81	100	22	25	3	4
10	100	121	25	30	5	4
15	225	256	48	54	6	6
20	400	441	78	85	7	7
25	625	676	114	122	8	8
30	900	961	154	162	8	9
40	1600	1681	251	263	12	11
50	2500	2601	367	378	11	13
60	3600	3721	503	519	16	15
70	4900	5041	654	668	14	17
90	8100	8281	1018	1038	20	20
99	9801	10000	1208	1229	21	22

Figure 2 shows the behavior of the real values of $\boldsymbol{\Pi_l(n)}$ in relation to the function $\boldsymbol{\Pi_l(n)}$ calculated with data taken from Table 3.

Taking into account that in 1852 Tschebycheff [20] published in his work "*Mémoire sur les nombres premiers,*" the demonstration that *Π(x)/(x/ln x)* for big *x* was of:

$$0,92129 \leq \frac{\Pi(x)}{\frac{x}{ln\ x}} \leq 1{,}10555 \qquad (6)$$

And in 1892 Sylvester [21] improved the above demonstration showing that the limit established for Tschebycheff for *p(x)/(x/ln x)* was of:

[3] From latín - Quad Eran Demonstrandum -

$$0{,}956 \leq \frac{\Pi(x)}{\frac{x}{\ln x}} \leq 1{,}045 \qquad (7)$$

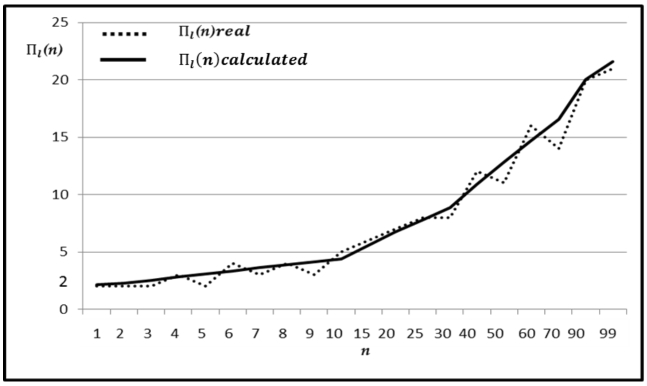

Fig2. *Comparative curves of $\Pi_l(n)$ real vs. $\Pi_l(n)$ calculated. Data taken from Table 3*

It is necessary to take into account those limits when applying the function *Π(x)*:

$$0{,}956 \leq \frac{\Pi_l(n) calculated}{\Pi_l(n)\ real} \leq 1{,}045 \qquad (8)$$

Inverting the above inequality:

$$1{,}046025 \geq \frac{\boldsymbol{\Pi_l(n) real}}{\boldsymbol{\Pi_l(n) calculated}} \geq 0{,}956938 \text{ for big x} \qquad (9)$$

Figure 3 shows the relationship using equation (9).

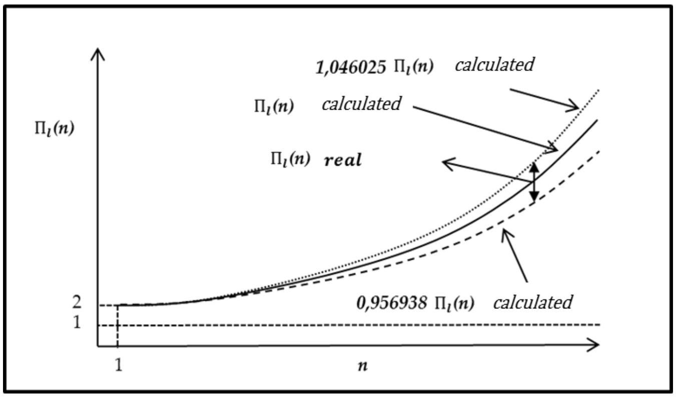

Fig3. *Graphical representation of equation (9).*

Given proof and demonstrating the Legendre Conjecture as $\infty \geq \boldsymbol{\Pi_l(n) > 1}$.

2.1. Solution to the Conjecture on the Existence of Infinite Prime Numbers of the Form $a^2 + 1$

Given the fact that there are eight columns of the 30-column array in [17] ordering procedure where prime numbers appear, the solution to this conjecture should be made for each column independently to proof that for integer values of n and a, it is necessary to verify where the following equation is true:

$$\begin{bmatrix} 30n+1 \text{ for } n \geq 1 \\ 30n+7 \text{ for } n \geq 0 \\ 30n+11 \text{ for } n \geq 0 \\ 30n+13 \text{ for } n \geq 0 \\ 30n+17 \text{ for } n \geq 0 \\ 30n+19 \text{ for } n \geq 0 \\ 30n+23 \text{ for } n \geq 0 \\ 30n+29 \text{ for } n \geq 0 \end{bmatrix} = a^2 + 1 \qquad (10)$$

2.2. Solution for the Primes of the form $\boldsymbol{30n + 1 = a^2 + 1},\ \boldsymbol{n \geq 1}$

Equation $30n + 1 = a^2 + 1$ has solutions for $(a, n) \in Z^+ - \{0\}$ in $30n = a^2$ where $n = 30^k$ and $a = 30^{\frac{k+1}{2}}$, k being an odd number.

Proof:

- $30n = a^2$
- When $n = 30, a = 30$, which indicate that the base of a is 30
- $30(30^k) = 30^{k+1} = a^2$ for $k > 0$ odd
- $a = 30^{\frac{k+1}{2}}$, as k is odd, there will always be a positive integer solution for a. Given 2 and 3.
- As infinite solutions of prime numbers of the form $30n + 1$ exist, there must also exist infinite solutions for primes of the form $30n + 1 \equiv 30^{k+1} + 1 \equiv a^2 + 1$ for k odd. Where $n = 30^k$ and $a = 30^{\frac{k+1}{2}}$.

Q.E.D.

For example: in the prime number680490000000001 $= 30n + 1 = a^2 + 1$, for$k = 9$, $n = 30^k = 30^9$, $a = 30^{\frac{k+1}{2}} = 30^5$ and $a^2 = 30^{k+1} = 30^{10}$.

2.3. Solution for Primes of the form $30n + 7 = a^2 + 1$

Equation $30n + 7 = a^2 + 1$ has solutions for $(a, n) \in Z^+ - \{0\}$ in $n = \frac{6^k - 6}{30}$ where $a = 6^{\frac{k}{2}}$ if k is an even number.

Proof:

- $30n + 7 = a^2 + 1$
- $30n + 6 = a^2$
- $6(5n + 1) = a^2$ therefore $6 \mid a^2$ and $(5n + 1) \mid a^2$, which indicate that for positive integer solutions to exist, a should be $a = 6^{\frac{k}{2}}$ for k even.
- $n = \frac{a^2 - 6}{30} = \frac{6^k - 6}{30}$ …………………………………………………………………… Given 2 and 3.
- As there are infinite solutions for primes of the form $30n + 7$, there must exist infinite solutions for primes of the form $30n + 7 \equiv 6^k + 1 \equiv a^2 + 1$ for even k where $n = \frac{6^k - 6}{30}$ and $a = 6^{\frac{k}{2}}$.

Q.E.D.

Example: in the prime number37 $= 30n + 7 = a^2 + 1$, for$k = 2$,$n = \frac{6^k - 6}{30} = \frac{6^2 - 6}{30} = 1$, $a = 6^{\frac{k}{2}} = 6$, $a^k = 6^2 = 36$.

2.4. Solution for Primes of the form $30n + 11 = a^2 + 1$

Equation $30n + 11 = a^2 + 1$ has solutions for $(a, n) \in Z^+ - \{0\}$ in $n = \frac{10^k - 10}{30}$ where $a = 10^{\frac{k}{2}}$ k is even.

Proof:

- $30n + 11 = a^2 + 1$
- $30n + 10 = a^2$
- $10(3n + 1) = a^2$, meaning $10 \mid a^2$ and $(3n + 1) \mid a^2$, which indicate that for positive integer solutions to exist, $a = 10^{\frac{k}{2}}$ for even$k$.
- $n = \frac{a^2 - 10}{30} = \frac{10^k - 10}{30}$ ………………………………………………………….… given 2 and 3.
- As there are infinite solutions for primes of the form $30n + 11$, there must exist infinite solutions for primes of the form $30n + 11 \equiv 10^k + 1 \equiv a^2 + 1$, for even k, where $n = \frac{10^k - 10}{30}$ and $a = 10^{\frac{k}{2}}$.

Q.E.D.

Example: in the prime number$101 = 30n + 11 = a^2 + 1$, for$k = 2$,$n = \frac{10^k - 10}{30} = \frac{10^3 - 10}{30} = 3$,

$a = 10^{\frac{k}{2}} = 10$, $a^k = 10^k = 10^2 = 100$

2.5. Solution for Primes of the form $\mathbf{30n + 13 = a^2 + 1}$

Equation$30n + 13 = a^2 + 1$, has no solutions for$(a, n) \in Z^+ + \{0\}$.

Proof:

- $30n + 13 = a^2 + 1$
- $30n + 12 = a^2$
- $n = \frac{a^2 - 12}{30}$, there is not any number $a \in Z^+$, that when squared ends in 2, then $n \neq\in Z^+ + \{0\}$.
- The equation $30n + 13 = a^2 + 1$ does not have solutions for $(a, n) \in Z^+ + \{0\}$ …. Given 3.

Q.E.D.

2.6. Solution for Primes of the form $\mathbf{30n + 17 = a^2 + 1}$

Equation $30n + 17 = a^2 + 1$ has positive integer solutions for $a \in Z^+$ and $n \in Z^+ + \{0\}$ in $n = \frac{4^k - 16}{30}$, where $a = 4^{\frac{k}{2}}$, for even k.

Proof:

- $30n + 17 = a^2 + 1$
- $30n + 16 = a^2$
- When$n = 0, a = 4$, which indicate the base of a is 4, when$n = 8, a = 16$, which can be expressed in general form as $a = 4^{\frac{k}{2}}$ for even$k$.
- $n = \frac{a^2 - 16}{30} = \frac{4^k - 16}{30}$ ………………………………………………………..….. Given 2 and 3.
- As there are infinite solutions for primes of the form $30n + 17$, there must exist infinite solutions for primes of the form $30n + 17 \equiv 4^k + 1 \equiv a^2 + 1$, for even k, where $n = \frac{4^k - 16}{30}$ and $a = 4^{\frac{k}{2}}$.

Q.E.D.

Example: in the prime number$17 = 30n + 1 = a^2 + 1$, for$k = 2$, $n = \frac{4^k - 16}{30} = \frac{16 - 16}{30} = 0$, $a = 4^{\frac{k}{2}} = 4$, $a^k = 4^2 = 16$.

2.7. Solution for Primes of the form $\mathbf{30n + 19 = a^2 + 1}$

Equation $30n + 19 = a^2 + 1$, has no solutions for $(a, n) \in Z^+ + \{0\}$

Proof:

- $30n + 19 = a^2 + 1$
- $30n + 18 = a^2$
- $n = \frac{a^2 - 18}{30}$, if there is not any number $a \in Z^+$, that when squared ends in 8, then $n \neq\in Z^+ + \{0\}$.
- The equation $30n + 19 = a^2 + 1$ has no solutions for $(a, n) \in Z^+ + \{0\}$ …. Given 3.

Q.E.D.

2.8. Solution for Primes of the form $\mathbf{30n + 23 = a^2 + 1}$

Equation $30n + 23 = a^2 + 1$ has no solutions for $(a, n) \in Z^+ - \{0\}$

Proof:

- $30n + 23 = a^2 + 1$
- $30n + 22 = a^2$
- $n = \frac{a^2-22}{30}$, if there is not any number $a \in Z^+$, that when squared ends in 2, then $n \neq\in Z^+ + \{0\}$.
- The equation $30n + 23 = a^2 + 1$, has no solutions for $(a, n) \in Z^+ + \{0\}$ …. Given 3.

Q.E.D.

2.9. Solution for Primes of the form $\mathbf{30n + 29 = a^2 + 1}$

Equation $30n + 29 = a^2 + 1$ has no solutions for $(a, n) \in Z^+ - \{0\}$

Proof:

- $30n + 29 = a^2 + 1$
- $30n + 28 = a^2$
- $n = \frac{a^2-28}{30}$, if there is not any number $a \in Z^+$, that when squared ends in 8, then $n \neq\in Z^+ + \{0\}$.
- The equation $30n + 23 = a^2 + 1$ has no solutions for $(a, n) \in Z^+ + \{0\}$ …. Given 3.

Q.E.D.

In summary, prime numbers of the form $[1, 7, 11, 17] + 30n \equiv a^2 + 1$, have solutions for $a \in Z^+ - \{0\}$, for $n \in Z^+ + \{0\}$, depending on which column primes form with n, where n has exponential progression according to 2.3.1, 2.3.2, 2.3.3 and 2.3.5, while primes of the form $[13, 19, 23, 29] + 30n \not\equiv a^2 + 1$, and has no solutions for $(a, n) \in Z^+ + \{0\}$, according to 2.3.4, 2.3.6, 2.3.7 and 2.3.8. Then if p_n is the n-th prime of the form $[1, 7, 11, 17] + 30n$ for:

$$n = \begin{bmatrix} 30^k & \text{odd } k \\ \dfrac{6^k - 6}{30} & \text{even } k \\ \dfrac{10^k - 10}{30} & \text{even } k \\ \dfrac{4^k - 16}{30} & \text{even } k \end{bmatrix} \text{ and } a = \begin{bmatrix} 30^{\frac{k+1}{2}} & \text{odd } k \\ 6^{\frac{k}{2}} & \text{even } k \\ 10^{\frac{k}{2}} & \text{even } k \\ 4^{\frac{k}{2}} & \text{even } k \end{bmatrix}$$

Then:

$$\lim_{n\to\infty} inf\,(p_n) = a^2 + 1 \qquad (11)$$

Furthermore, a similar expression to Equation (11) is identical using Modular Identities:

$$p_n \equiv \begin{bmatrix} 1 \\ 7 \\ 11 \\ 13 \end{bmatrix} mod\ 30 \text{ for } n = \begin{bmatrix} 30^k & \text{odd } k \\ \frac{6^k-6}{30} & \text{even } k \\ \frac{10^k-10}{30} & \text{even } k \\ \frac{4^k-16}{30} & \text{even } k \end{bmatrix} \text{ and}$$

$$a^2 + 1 \equiv \begin{bmatrix} 1 \\ 7 \\ 11 \\ 13 \end{bmatrix} mod\ 30 \text{ for } a = \begin{bmatrix} 30^{\frac{k+1}{2}} & \text{odd } k \\ 6^{\frac{k}{2}} & \text{even } k \\ 10^{\frac{k}{2}} & \text{even } k \\ 4^{\frac{k}{2}} & \text{even } k \end{bmatrix}$$

Q.E.D.

Note: In [17] it was proved that the cell n_1 where a prime p_n is formed, the following cells $n = n_1 + kp_n$ for $k \geq 1$ have no primes. Here n has arithmetic progression that is different from geometric progression in 2.3.1, 2.3.2, 2.3.3 and 2.3.5 for n and must have cells where both do not have coincidence, therefore $p_n = a^2 + 1$ as $n \rightarrow \infty$ must exist.

3. The Solution to Other Conjectures

3.1. Solution to Conjecture 1379

Conjecture 1379 was presented by Porras-Ferreira in 2012 [7], *"There are infinite primes that end in 1, 3, 7 and 9 in a consecutive way,"* during the 5º. International Congress of Mathematics in Bogotá, Colombia and can be solved in the following form:

Rows 0, 3 and 6 of Table 1 in Table 4 shows the nature of prime numbers that end in 1, 3, 7 y 9 in a consecutive way in the same row, Origin of 1379 conjecture.

Table4. *Location of the prime numbers that end in 1, 3, 7 and 9 in a consecutive way in the same row, Origin of 1379 conjecture.*

Rows n	Column 11	Column 13	Column 17	Column 19
0	11	13	17	19
3	101	103	107	109
6	191	193	197	199

In other words, the conjecture only occurs in the rows:

$$n = \begin{bmatrix} 0 + 7k \\ 3 + 7k \\ 6 + 7k \end{bmatrix} \text{ for } k \geq 0 \tag{12}$$

and in the columns 11, 13, 17 y 19 where prime numbers form consecutively.

Given that within every column of Prime Numbers Order Array are infinite, it can be concluded that prime numbers that form in the rows in Equation (12) and simultaneously in columns 11, 13, 17, 19 are also infinite taken into account that Equation (12) replicates from $k = 0$ to infinity, existing columns 11, 13, 17 and 19 where they coincide and simultaneous primes form.

Several examples are given in Table 5 where the primes that comply with Conjecture 1379 are calculated:

Table5. *Examples of prime numbers that comply the Conjecture 1379 to infinity according to equation (9).*

Rows			Primes			
n		k	Column 11	Column 13	Column 17	Column 19
	0	0	11	13	17	19
	49	7	1481	1483	1487	1489
0+7k	189	27	5681	5683	5687	5689
	1757	251	52721	52723	52727	52729
	….	….	….1	….3	….7	….9
	3	0	101	103	107	109
3+7k	244674	34953	7340231	7340233	7340237	7340239
	32059758	4579965	961792751	961792753	961792757	961792759
	….	….	….1	….3	….7	….9
	6	0	191	193	197	199
	27	3	821	823	827	829
6+7k	69	9	2081	2083	2087	2089
	….	….	….1	….3	….7	….9

3.2. Solution to Brocard's Conjecture

Brocard's Conjecture says that "There are at least four prime numbers between $(p_n)^2$ and $(p_{n+1})^2$, for **n>1**, where p_n is the *n*- prime number [8]."

The amount of prime numbers in between the squares of consecutive primes is 2, 5, 6, 15, 9, 22, 11, 27, ….. from that, it can be said that Brocard's Conjecture establishes that at least four prime numbers exist in between the squares of two consecutive primes greater than 2.

The smallest distance between two consecutive prime numbers is 2, which corresponds to the twin prime numbers, so this conjecture can be restricted even more in the sense that it will always be at least four primes between the squares of two numbers *n* and *n+2*, for *n*≥ 3 and $n \in Z^{+}$**.** If this can be proven, Brocard's Conjecture will be proven, because if four primes exist between the squares of these numbers, there should be at least four primes when the separation between two consecutive primes is greater than 2, as to say between the squares of *n* and *n+a* for *a>2*.

Expressed in mathematical form:

$\Pi_b(n)\ \geq 4$ between n^2 and $(n+2)^2$ where $\Pi_b(n)$ is the amount of prime numbers in between the two squares, in other words:

$\Pi_b(n) = \Pi((n+2)^2) - \Pi(n^2) \geq 4$

Where $\boldsymbol{\Pi}((\boldsymbol{n}+\mathbf{2})^{\mathbf{2}})$ and $\Pi(n^2)$ represent the amount of prime numbers between $(n+2)^2$ and n^2 respectively and $\Pi_b(n)$ the amount of primes in the difference of both functions.

3.2.1. Theorem that solves Brocard's Conjecture

There will always exist at least four prime numbers in between $\boldsymbol{p_n}^{\mathbf{2}}$ and $\boldsymbol{p_{n+1}}^{\mathbf{2}}$**,** for ***n*>*1*, *p*** ∈ ***P*** as to say:

$$\Pi_b(n) = \Pi((n+2)^2) - \Pi(n^2) \cong \frac{2n+2}{ln\ (n+2)} \geq 4 \tag{13}$$

- The **theorem of the prime numbers** establishes that the amount of prime numbers less than x for a very big x is:

$\Pi(x) \cong \frac{x}{ln\ x}$

- Therefore:

$\Pi(n^2) \cong \frac{n^2}{ln\ n^2}$ and $\Pi((n+2)^2) \cong \frac{(n+2)^2}{ln\ (n+2)^2}$

$\Pi_b(n) = \Pi(n+2)^2 - \Pi(n^2) \cong \frac{(n+2)^2}{ln\ (n+2)^2} - \frac{n^2}{ln\ n^2}$

………………………………………………………………………………………… given 1.

- $\Pi_b(n) \cong \frac{n^2}{ln\,(n+2)^2} - \frac{n^2}{ln\ n^2} + \frac{4n+4}{ln\ (n+2)^2}$
- Dividing both terms by $\frac{2n+2}{\ln(n+2)}$:

$$\left(\frac{\Pi_l(n)}{\left(\frac{2n+2}{\ln(n+2)}\right)}\right) \cong \left(\frac{\frac{n^2}{\ln(n+2)^2} - \frac{n^2}{\ln n^2}}{\left(\frac{2n+2}{\ln(n+2)}\right)}\right) + 1$$

- Applying limits to both functions there can be established that:

$$\lim_{n\to\infty}\left(\frac{\Pi_l(n)}{\left(\frac{2n+2}{\ln(n+2}\right)}\right) = \lim_{n\to\infty}\left(\frac{\frac{n^2}{\ln(n+2)^2} - \frac{n^2}{\ln n^2}}{\left(\frac{2n+2}{\ln(n+2)}\right)}\right) + 1 = 0 + 1 = 1$$

Note: the numerator (first term) of the right-hand side of the equation tends to grow very slowly, while the denominator tends to grow towards infinite more quickly.

And

$$\lim_{n\to\infty} \Pi_b(n)\ = \lim_{n\to\infty}\left(\frac{2n+2}{ln\ (n+2)}\right) = \infty$$

- $\boldsymbol{\Pi_b(n)}$ is an ascending, continuous, divergent function since it has no limits, …………………………………………………………………………..……….. given 4.

- Therefore it can be said that:

$\Pi_b(n) \cong \frac{2n+2}{ln\ (n+2)}$ ………………………… given 4 and 5.

- And if $\boldsymbol{\Pi_b(n) \geq 4}$ for $\boldsymbol{n \geq 3}$ can be demonstrated, Brocard's Conjecture can also be demonstrated.
- Verifying the above for **n=3**

$$\Pi_b \cong \frac{2*3+2}{\ln(3+2)} = \frac{8}{\ln 5} = 4{,}97 > 4$$

Q.E.D.

Table 6 shows the verification of this function with relation to real calculations of the amount of primes between $\boldsymbol{n^2}$ and $\boldsymbol{(n+2)^2}$.

Table6. *Calculation of* $\boldsymbol{\Pi_b(n)}$ ***real*** *.vs.* $\boldsymbol{\Pi_b(n)}$ ***calculated.***

n	n^2	$(n+2)^2$	$\Pi(n^2)$	$\Pi((n+2)^2)$	$\Pi_b(n)real$	$\Pi_b(n)calculated$
3	9	25	4	9	5	5
4	16	36	6	11	5	6
5	25	49	9	15	6	6
6	36	64	11	18	7	7
7	49	81	15	22	7	7
8	64	100	18	25	7	8
9	81	121	22	30	8	8
10	100	144	25	34	9	9
11	121	169	30	39	9	9
12	144	196	34	44	10	10
13	169	225	39	48	9	10
14	196	256	44	54	10	11
15	225	289	48	61	13	11
20	400	484	78	92	14	14
30	900	1024	154	172	18	18
40	1600	1764	251	275	24	22
50	2500	2704	367	393	26	26
60	3600	3844	503	532	29	30
70	4900	5184	654	690	36	33
80	6400	6724	834	867	33	37
90	8100	8464	1018	1058	40	40
98	9604	10000	1185	1229	44	43

Figure 4 shows the behavior of the real values of *Π$_b$(n)* with respect to the results of the function *Π$_b$(n)*. Data taken from Table 6.

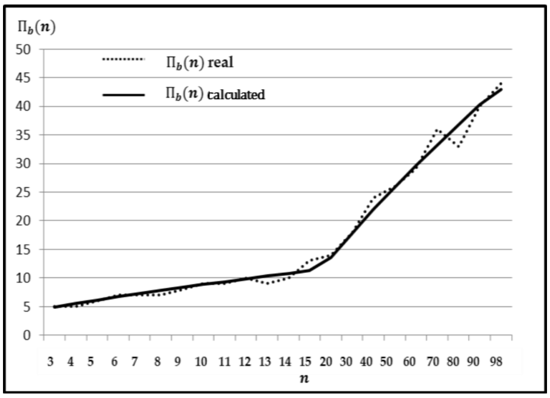

Fig4. *Comparative curves of* $\boldsymbol{\Pi_b(n)}$ ***real*** *.vs.* $\boldsymbol{\Pi_b(n)}$ ***calculated.***

Taking into account that in 1852 Tschebycheff [20] published the demonstration that ***p(x)/(x/ln x)*** for a very big ***x***

$0,92129 \leq \frac{\Pi(x)}{\frac{x}{ln\ x}} \leq 1{,}10555$ (14)

In 1892 Sylvester [21] improved the above demonstration showing that the limit established by Tschebycheff for ***Π(x)/(x/ln x)*** was of:

$$0{,}956 \leq \frac{\Pi(x)}{\frac{x}{\ln x}} \leq 1{,}045 \qquad (15)$$

When applied to the function $\Pi(x) \cong \frac{x}{\ln x}$ it is necessary to observe those limits, therefore:

$$0{,}956 \leq \frac{\Pi_b(n)calculated}{\Pi_b(n)real} \leq 1{,}045 \qquad (16)$$

If the above inequity is reversed:

$$1{,}046025 \leq \frac{\Pi_b(n)real}{\Pi_b(n)calculated} \geq 0{,}956938 \quad \text{for very big } \boldsymbol{x} \qquad (17)$$

Figure 5 shows the behavior of the real values of $\Pi_b(n)$ with respect to the results of the function $\Pi_b(n)$ according to Equation (17).

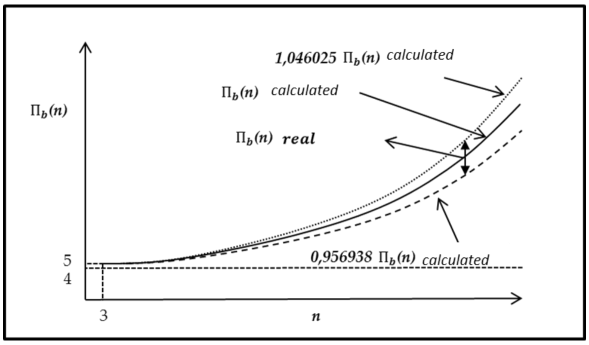

Fig5. *Graphical representation of equation (17)*

In this manner, the proof is finished, demonstrating Brocard's Conjecture as true since $\infty \geq \boldsymbol{\Pi_b(n)} > 4$, raising this conjecture to the category of theorem.

4. Conclusion

The finding of a Prime Numbers Order Array in Porras-Ferreira and Andrade [17] facilitates the solution of many conjectures related to the primes including the ones that appeared as "unattackable at the present state of science" of Landau´s problems.

Using this found order, Modular Identities and Residue System, the solution to the Conjecture of the Twin Prime Numbers, and the Conjecture on the existence of infinite prime numbers p, such as p-1, is a perfect square are presented. Likewise, the solution to the Conjecture 1379 were found and demonstrated. The exact accomplishment of each one of the conjectures to the Prime Numbers Order found in [17] confirms the truthfulness of it. Also, true solutions for Legendre's Conjecture and Brocard's Conjecture were found and elevated to theorems.

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References

[1] CURBERA G. ICM through history. *Newsletter of the European Mathematical Society*, *63*, pp. 16-21, March, 2007.

[2] HARDY, G. H. and LITTLEWOOD, J. E. "Some Problems of 'Partitio Numerorum.' III. On the Expression of a Number as a Sum of Primes." *Acta Math. 44*, 1-70, 1923.

[3] GOLDBACH, C. Letter to L. Euler, June 7, 1742.

[4] CHEN, J. R. On the Distribution of Almost Primes in an Interval, *Sci. Sinica 18*, 611-627, 1975.

[5] HARDY, G. H. and WRIGHT, W. M. "Unsolved Problems Concerning Primes." §2.8 and Appendix §3 in An Introduction to the Theory of Numbers, 5th ed. Oxford, England: Oxford University Press, pp. 19 and 415-416, 1979.

[6] EULER, L, Novi Commentarii academiae scientiarum Petropolitanae 9, 1764, pp. 99-153. Reprinted in Commentat. arithm. 1, 356-378, 1849. Reprinted in Opera Omnia: Series 1, Volume 3, pp. 1-45.

[7] PORRAS-FERREIRA, W., An approach to the solution of the Goldbach's conjecture and Fermat's Theorem (Spanish) Sergio Arboleda University. 5th Congress of mathematics, Bogota, Colombia 16-18 June, 2013.

[8] WELLS, D., Prime numbers: The most mysterious figures in math, p.22, John Wiley & Sons, Inc., 2005.

[9] PINTZ, J., on primes in short intervals, I, *Studia Sci. Math. Hungar. 16*, 395–414, 1981.

[10] PINTZ, J., on primes in short intervals, II, *Studia Sci. Math. Hungar. 19*, 89–96, 1984.

[11] WATT, N., Short intervals almost all containing primes, *Acta Arith. 72*, 131–167, 1995.

[12] PINTZ, J., Very large gaps between consecutive primes, *J. Number Th. 63* 286–301, 1997.

[13] BAKER, R.C., HARMAN, G., PINTZ J., The difference between consecutive primes II, *Proc. London Math. Soc. (3) 83*, 532-562, 2001.

[14] GOLDSTON, D. A., MOTOHASHI, Y., PINTZ, J., YILDIRIM, C. Y., Small gaps between primes exist, *Proc. Japan Acad. 82A*, 61–65, 2006.

[15] PERELLI, A., PINTZ, J., on the exceptional set for the 2k-twin prime's problem, *Compositio Math. 82, No. 3*, 355–372, 1992.

[16] ZHANG, Y., Bounded gaps between primes, *Annals of Mathematics, 179, 1121-1174*, 2014.

[17] PORRAS-FERREIRA J.W. and ANDRADE C., The formation of prime numbers and the solution for Goldbach's conjectures. *World Open Journal of Advanced Mathematics. Vol* 2 , No 1, pp 1-32, 2014. Available online at: http://scitecpub.com/journals.php.

[18] TAO, T., Polymath8, Bounded gaps between primes, The Polymath blog, http:www.polymath projects.org, 2014.

[19] MAYNARD, J., Small gaps between primes, Cornell University Library, arXiv: 1311.4600, 2013.

[20] TSCHEVICHEFF, P., Mémoire sur les nombres premiers. Journal de Mathématiques Pures et Appliquées, 1re série, tome 17 p. 366-390, 1852.

[21] SYLVESTER, J.J. On Tchebycheff's theorem of the totality of prime numbers comprised within given limits, American Journal of Mathematics N° 4 pp. 230–247, 1881.