

On Some Multidimensional Fractional Integral Operators Involving Multivariable A -Function as Kernel

Nanda Kulkarni*

Department of Mathematics,
 Maharani Lakshmi Ammanni College for Women Bangalore, Karnataka, India

*Research Scholar: Department of Mathematics,
 J.J.T.University, Chudela, Jhunjhunu (Rajasthan), India
nandavnadiger@gmail.com

Yashwant Singh

Department of Mathematics,
 Government College, Kaladera, Jaipur, Rajasthan, India
dryashu23@yahoo.in

Abstract: *In the present paper, we study certain multidimensional fractional integral operators involving a general A -function in their kernel. We give five basic properties of these operators, and then establish two theorems and two corollaries, which are believed to be new. These basic theorems exhibit structural relationships between the multidimensional integral transforms. The one-and –two –dimensional analogues of these results, which are new and of interest in themselves, can easily be deduced. Special cases of these latter theorems will give rise to certain known results obtained from time to time by several earlier authors.*

Keywords: *Fractional Integral Operators, Multivariable A -function, Multidimensional Integral Transforms.*

1. INTRODUCTION

Fractional integral operators have been defined and studied by various authors notably by Riemann and Liouville [4], Weyl [4], Erdelyi [1,2], Kober [8], Sneddon [15], Kalla [11], Saxena [13], Srivastava and Buschmann [16] etc.. These operators play an important role in the theory of integral equations and problems concerning Mathematical Physics. In this paper, we shall study the following multidimensional fractional integral operators having the general function ϕ as their kernel. Also, for the sake of brevity, we shall use the symbol $f(x)$ to represent $f(x_1, x_2, \dots, x_r)$.

Gautam and Goyal [5] defined and represented the multivariable A -function as follows:

$$\begin{aligned}
 A[z_1, \dots, z_r] &= A_{p, q; p_1, q_1; \dots; p_r, q_r}^{m, n; m_1, n_1; \dots; m_r, n_r} \\
 &\left[\begin{array}{c} z_1 \\ \vdots \\ z_r \end{array} \middle| \begin{array}{c} (a_j; A_j, \dots, A_j^{(r)})_{1, p}; (c_j, C_j)_{1, p_1}; \dots; (c_j^{(r)}, C_j^{(r)})_{1, p_r} \\ (b_j; B_j, \dots, B_j^{(r)})_{1, q}; (d_j, D_j)_{1, q_1}; \dots; (d_j^{(r)}, D_j^{(r)})_{1, q_r} \end{array} \right] \\
 &= \frac{1}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} \theta_1(s_1) \dots \theta_r(s_r) \Phi(s_1, \dots, s_r) z_1^{s_1} \dots z_r^{s_r} . ds_1 \dots ds_r \quad (1.1)
 \end{aligned}$$

Where $\omega = \sqrt{-1}$;

$$\theta_i(s_i) = \frac{\prod_{j=1}^{m_i} \Gamma(d_j^{(i)} - D_j^{(i)} s_i) \prod_{j=1}^{n_i} \Gamma(1 - c_j^{(i)} + C_j^{(i)} s_i)}{\prod_{j=m_i+1}^{q_i} \Gamma(1 - d_j^{(i)} + D_j^{(i)} s_i) \prod_{j=n_i+1}^{p_i} \Gamma(c_j^{(i)} - C_j^{(i)} s_i)}$$

$$\forall i \in \{1, \dots, r\} \tag{1.2}$$

$$\Phi(s_1, \dots, s_r) = \frac{\prod_{j=1}^n \Gamma(1 - a_j + \sum_{i=1}^r A_j^{(i)} s_i) \prod_{j=1}^m \Gamma(b_j - \sum_{i=1}^r B_j^{(i)} s_i)}{\prod_{j=n+1}^p \Gamma(a_j - \sum_{i=1}^r A_j^{(i)} s_i) \prod_{j=m+1}^q \Gamma(1 - b_j + \sum_{i=1}^r B_j^{(i)} s_i)} \tag{1.3}$$

Here $m, n, p, q, m_i, n_i, p_i,$ and $q_i (i = 1, \dots, r)$ are non-negative integers and all $a_j, b_j, d_j^{(i)}, s, c_j^{(i)}, s, A_j^{(i)}, s$ and $B_j^{(i)}, s$ are complex numbers.

The multiple integral defining the A -function of r -variables converges absolutely if

$$\left| \arg(\Omega_i) z_k \right| < \frac{\pi}{2} \eta_i, \xi_i^* = 0, \eta_i > 0 \tag{1.4}$$

$$\Omega_i = \prod_{j=1}^p \{A_j^{(i)}\}^{A_j^{(i)}} \prod_{j=1}^q \{B_j^{(i)}\}^{-B_j^{(i)}} \prod_{j=1}^{q_i} \{D_j^{(i)}\}^{D_j^{(i)}} \cdot \prod_{j=1}^{p_i} \{C_j^{(i)}\}^{-C_j^{(i)}} \\ , \forall i \in \{1, \dots, r\}; \tag{1.5}$$

$$\xi_i^* = I_m \left[\sum_{j=1}^p A_j^{(i)} - \sum_{j=1}^q B_j^{(i)} + \sum_{j=1}^{q_i} D_j^{(i)} - \sum_{j=1}^{p_i} C_j^{(i)} \right], \forall i \in \{1, \dots, r\} \tag{1.6}$$

$$\eta_i = \operatorname{Re} \left[\sum_{j=1}^n A_j^{(i)} - \sum_{j=n+1}^p A_j^{(i)} + \sum_{j=1}^m B_j^{(i)} - \sum_{j=m+1}^q B_j^{(i)} + \sum_{j=1}^{m_i} D_j^{(i)} - \sum_{j=m_i+1}^{q_i} D_j^{(i)} + \sum_{j=1}^{n_i} C_j^{(i)} - \sum_{j=n_i+1}^{p_i} C_j^{(i)} \right] \\ \forall i \in \{1, \dots, r\}; \tag{1.7}$$

If we take A_j, B_j, C_j, s and D_j, s as real and positive and $m = 0$, the A -function reduces to multivariable H -function of Srivastava and Panda [17]

We are using the multivariable A -function in the following concise form throughout the text.

$$A[z_1, \dots, z_r] = A_{p,q;N_r}^{m,n;M_r} \left[\begin{matrix} z_1 \\ P : P_r^{(r)} \\ \vdots \\ Q : Q_r^{(r)} \\ z_r \end{matrix} \right] \\ = \frac{1}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} \theta_1(s_1) \dots \theta_r(s_r) \Phi(s_1, \dots, s_r) z_1^{s_1} \dots z_r^{s_r} ds_1 \dots ds_r \tag{1.8}$$

Where $\omega = \sqrt{-1}$;

$$M_r = m_1, n_1; \dots; m_r, n_r;$$

$$N_r = p_1, q_1; \dots; p_r, q_r;$$

$$P = (a_j; A_j, \dots, A_j)_{1,p}; \quad Q = (b_j; B_j, \dots, B_j)_{1,q};$$

$$P_r^{(r)} = (c_j, C_j)_{1,p_1}; \dots; (c_j, C_j)_{1,p_r};$$

And the definition of the functions $\theta_i(s_i) i = 1, \dots, r; \Phi(s_1, \dots, s_r)$ and the condition of existence of the multivariable A -function are the same as mentioned by Gautam and Goyal [5].

We introduce the fractional integration operators by means of the following integral equations:

$$Y \{ f(x) \} = Y \{ f(x); t_1, \dots, t_r; \gamma_1, \dots, \gamma_r \} = \prod_{i=1}^r (t_i^{\gamma_i-1}) \int_0^{t_1} \dots \int_0^{t_r} \prod_{i=1}^r \left\{ (x_i)^{\gamma_i} \phi \left(\frac{x_1}{t_1}, \dots, \frac{x_r}{t_r} \right) \right\} f(x) dx_1 \dots dx_r \quad (1.9)$$

And

$$N \{ f(x) \} = N \{ f(x); t_1, \dots, t_r; \delta_1, \dots, \delta_r \} = \prod_{i=1}^r (t_i^{\delta_i}) \int_{t_1}^{\infty} \dots \int_{t_r}^{\infty} \prod_{i=1}^r \left\{ (x_i)^{-\delta_i-1} \phi \left(\frac{t_1}{x_1}, \dots, \frac{t_r}{x_r} \right) \right\} f(x) dx_1 \dots dx_r \quad (1.10)$$

Where the kernel ϕ is such that the above integrals make sense. The above operators exist under the following conditions:

- (i) $p_i \leq 1, q_i < \infty, \frac{1}{p_i} + \frac{1}{q_i} = 1, i = 1, 2, \dots, r$
- (ii) $\text{Re}(\gamma_i) > -\frac{1}{q_i}; \text{Re}(\delta_i) > -\frac{1}{p_i}; i = 1, 2, \dots, r$
- (iii) $f(x) \in L_{p_i}((0, \infty), \dots, (0, \infty)); i = 1, 2, \dots, r$
- (iv) $0 \leq m_i \leq q_i, 0 \leq n_i \leq p_i, q_k \geq 0, 0 \leq n_k \leq p_k$

The following special case of the multidimensional fractional integral operators involving product of Gauss's hypergeometric functions ([14], p.153, eq. (i) and (ii)) will be used in Section 3.

$$I \{ f(x) \} = \prod_{i=1}^r \left\{ \frac{t_i^{\gamma_i-1}}{\Gamma(1-\alpha_i)} \right\} \int_0^{t_1} \dots \int_0^{t_r} \prod_{i=1}^r \left\{ (x_i)^{\gamma_i} {}_2F_1 \left(\alpha_i, \beta_i + m; \beta_i; \frac{x_i}{t_i} \right) \right\} f(x) dx_1 \dots dx_r \quad (1.11)$$

And

$$K \{ f(x) \} = \prod_{i=1}^r \left\{ \frac{t_i^{\delta_i}}{\Gamma(1-\alpha_i)} \right\} \int_{t_1}^{\infty} \dots \int_{t_r}^{\infty} \prod_{i=1}^r \left\{ (x_i)^{-\delta_i-1} {}_2F_1 \left(\alpha_i, \beta_i + m; \beta_i; \frac{t_i}{x_i} \right) \right\} f(x) dx_1 \dots dx_r \quad (1.12)$$

The conditions of existence of these operators follow easily from the conditions given in the paper referred to above.

The generalized multidimensional integral transform T , defined below, will also be required during the course of our study:

$$T \{ f(x); s_1, \dots, s_r \} = \int_0^{\infty} \dots \int_0^{\infty} k(s_1 x_1, \dots, s_r x_r) f(x) dx_1 \dots dx_r \quad (1.13)$$

Where $k(s_1 x_1, \dots, s_r x_r)$ is the kernel of the transform T and the multiple integral occurring in the equation (1.8) is assumed to be convergent.

The following multivariable A -function transform will also be used in the sequel:

$$A \{ f(x); s_1, \dots, s_r \} = \int_0^{\infty} \dots \int_0^{\infty} A_{p, q; p_1, q_1; \dots; p_r, q_r}^{m, n; m_1, n_1; \dots; m_r, n_r} f(x) dx_1 \dots dx_r$$

$$\int_{s_1 x_1}^{s_r x_r} \left[\begin{matrix} (a_j; A_j, \dots, A_j^{(r)})_{1,p}; (c_j, C_j)_{1,p_1}; \dots; (c_j^{(r)}, C_j^{(r)})_{1,p_r} \\ (b_j; B_j, \dots, B_j^{(r)})_{1,q}; (d_j, D_j)_{1,q_1}; \dots; (d_j^{(r)}, D_j^{(r)})_{1,q_r} \end{matrix} \right] f(x) dx_1 \dots dx_r \tag{1.14}$$

The transform defined above will be denoted symbolically as follows:

$$A \left\{ f(x); \begin{matrix} m, n, m_1, n_1, \dots; m_r, n_r; (a_j; A_j, \dots, A_j^{(r)})_{1,p}; (c_j, C_j)_{1,p_1}; \dots; (c_j^{(r)}, C_j^{(r)})_{1,p_r}; \\ p, q; p_1, q_1, \dots; p_r, q_r; (b_j; B_j, \dots, B_j^{(r)})_{1,q}; (d_j, D_j)_{1,q_1}; \dots; (d_j^{(r)}, D_j^{(r)})_{1,q_r}; \\ s_1 x_1 \\ s_r x_r \end{matrix} \right\} \tag{1.15}$$

2. SOME BASIC PROPERTIES

Property1. If $f(x) \in L_{p_i}((0, \infty), \dots, (0, \infty)); 1 \leq p_i \leq 2$ (or $f(x) \in M_{p_i}((0, \infty), \dots, (0, \infty)); p_i > 2$), where $M_{p_i}((0, \infty), \dots, (0, \infty))$ denotes the class of all functions $f(x) \in L_{p_i}((0, \infty), \dots, (0, \infty)); p_i > 2$, which are the inverse Mellin transforms of functions belonging to $L_{q_i}((-\infty, \infty), \dots, (-\infty, \infty)); \text{Re}(\gamma_i) > \frac{1}{q_i}, \text{Re}(\delta_i) > \frac{1}{p_i}, \frac{1}{p} + \frac{1}{q} = 1 (i = 1, 2, \dots, r)$ and the multi dimensional Mellin transform of the function $f(x)$ exists, then

$$(a) M \left[Y \{ f(x); t_1, \dots, t_r; \gamma_1, \dots, \gamma_r \}; s_1, \dots, s_r \right] = M \left[f(x); s_1, \dots, s_r \right] N \left\{ 1; t_1, \dots, t_r; \gamma_1 - s_1 + 1, \dots, \gamma_r - s_r + 1 \right\} \tag{2.1}$$

$$(b) M \left[N \{ f(x); t_1, \dots, t_r; \delta_1, \dots, \delta_r \}; s_1, \dots, s_r \right] = M \left[f(x); s_1, \dots, s_r \right] Y \left\{ 1; t_1, \dots, t_r; \delta_1 + s_1 + 1, \dots, \delta_r + s_r + 1 \right\} \tag{2.2}$$

Where the symbol M occurring in (2.1) and (2.2) stands for the multidimensional Mellin transform defined in the following way:

$$M \left\{ f(x); s_1, \dots, s_r \right\} = \int_0^\infty \dots \int_0^\infty f(x) \prod_{i=1}^r (x_i^{s_i-1}) dx_1 \dots dx_r \tag{2.3}$$

Provided that the multiple integral involved in (2.3) exists.

Proof: To prove (2.1), we use (2.3) and (1.9) to obtain

$$M \left[Y \{ f(x) \} \right] = \int_0^\infty \dots \int_0^\infty (t_i^{s_i-1}) \left\{ \int_0^{t_1} \dots \int_0^{t_r} \prod_{i=1}^r (x_i^{\gamma_i}) f(x) \phi \left(\frac{x_1}{t_1}, \dots, \frac{x_r}{t_r} \right) \right\} dt_1 \dots dt_r \tag{2.4}$$

Now interchanging the orders of t_i and $x_i (i = 1, 2, \dots, r)$ integrals in (2.4), which is easily seen to be permissible under the conditions stated with (2.1) and interpreting the results thus obtained with the help of (1.10), we arrive at the required result (2.1).

The result (2.2) can be established in a similar manner.

Property2. If $f(x) \in L_{p_i}((0, \infty), \dots, (0, \infty)); 1 \leq p_i \leq 2$, $g(x) \in L_{q_i}((0, \infty), \dots, (0, \infty));$,

$$\text{Re}(\gamma_i) > \max \left(-\frac{1}{p_i}, -\frac{1}{q_i} \right), (i = 1, 2, \dots, r), \text{ then}$$

$$\int_0^\infty \dots \int_0^\infty f(x) Y \{ g(x) \} dx_1 \dots dx_r = \int_0^\infty \dots \int_0^\infty g(x) N \{ f(x) \} dx_1 \dots dx_r \tag{2.5}$$

Provided that the multiple integrals involved in (2.5) are absolutely convergent.

Property3.

$$(a) Y \{ f (wx); t_1, \dots, t_r; \gamma_1, \dots, \gamma_r \} = Y \{ f (x); t_1 w_1, \dots, t_r w_r; \gamma_1, \dots, \gamma_r \} \tag{2.6}$$

$$(b) N \{ f (wx); t_1, \dots, t_r; \delta_1, \dots, \delta_r \} = W \{ f (x); t_1 w_1, \dots, t_r w_r; \delta_1, \dots, \delta_r \} \tag{2.7}$$

Provided that the multiple integrals involved in (2.6) and (2.7) are absolutely convergent.

Property4.

$$(a) Y \{ f (1 / x); t_1, \dots, t_r; \gamma_1, \dots, \gamma_r \} = Y \{ f (x); 1 / t_1, \dots, 1 / t_r; \gamma_1 + 1, \dots, \gamma_r + 1 \} \tag{2.8}$$

$$(b) N \{ f (1 / x); t_1, \dots, t_r; \delta_1, \dots, \delta_r \} = W \{ f (x); 1 / t_1, \dots, 1 / t_r; \delta_1 - 1, \dots, \delta_r - 1 \} \tag{2.9}$$

Provided that the multiple integrals involved in (2.8) and (2.9) are absolutely convergent.

Property5.

$$(a) Y \left\{ \prod_{i=1}^r (x_i^{\beta_i}) f (x); t_1, \dots, t_r; \gamma_1, \dots, \gamma_r \right\} = \prod_{i=1}^r (t_i^{\beta_i}) Y \{ f (x); t_1, \dots, t_r; \gamma_1 + \beta_1, \dots, \gamma_r + \beta_r \} \tag{2.10}$$

$$(b) N \left\{ \prod_{i=1}^r (x_i^{\beta_i}) f (x); t_1, \dots, t_r; \delta_1, \dots, \delta_r \right\} = \prod_{i=1}^r (t_i^{\beta_i}) N \{ f (x); t_1, \dots, t_r; \delta_1 - \beta_1, \dots, \delta_r - \beta_r \} \tag{2.11}$$

Provided that the multiple integrals involved in (2.10) and (2.11) are absolutely convergent.

The proofs of the properties 2 to 5 are straight forward and follow from the definitions (1.9) and (1.10) of the multidimensional fractional integral operators involved therein.

3. RELATIONSHIP BETWEEN MULTIDIMENSIONAL FRACTIONAL INTEGRAL OPERATORS AND MULTIDIMENSIONAL INTEGRAL TRANSFORMS

In this section, we shall establish two most general theorems exhibiting interconnections between the fractional integral operators Y and N defined by (1.9) and (1.10) respectively and the integral transform T defined by (1.13). Next, we give two interesting corollaries interconnecting the multidimensional fractional integral operators defined by (1.1) and (1.12) and the multidimensional A -function transform defined by (1.14).

Theorem1. If

$$\tau (s_1, \dots, s_r) = T \{ \psi (u^\rho) g (u); s_1, \dots, s_r \} = \int_0^\infty \dots \int_0^\infty k (su) \psi (u^\rho) g (u) du_1 \dots du_r \tag{3.1}$$

And

$$\psi (t_1, \dots, t_r) = Y \{ f (x^\sigma); t_1, \dots, t_r \} = \prod_{i=1}^r (x_i^{-\gamma_i - 1}) \int_0^{t_1} \dots \int_0^{t_r} f (x^\sigma) \prod_{i=1}^r (x_i^{\gamma_i}) \phi \left(\frac{x_1}{t_1}, \dots, \frac{x_r}{t_r} \right) dx_1 \dots dx_r \tag{3.2}$$

Then

$$\tau(s_1, \dots, s_r) = \frac{1}{\rho_1 \dots \rho_r} \int_0^\infty \dots \int_0^\infty f(x^\sigma) \theta(s_1, \dots, s_r, x_1, \dots, x_r, \rho_1, \dots, \rho_r) dx_1 \dots dx_r \tag{3.3}$$

Where

$$\theta(s_1, \dots, s_r, x_1, \dots, x_r, \rho_1, \dots, \rho_r) = Y \left\{ \prod_{i=1}^r \left(x_i^{\frac{1}{\rho_i}-1} \right) g \left(x^{\frac{1}{\rho}} \right) k \left(sx^{\frac{1}{\rho}} \right); t_1, \dots, t_r \right\} = \prod_{i=1}^r \left(t_i^{\gamma_i} \right) \int_{t_1}^\infty \dots \int_{t_r}^\infty \prod_{i=1}^r \left(x_i^{\frac{1}{\rho_i}-\gamma_i-2} \right) g \left(x^{\frac{1}{\rho}} \right) k \left(sx^{\frac{1}{\rho}} \right) \phi \left(\frac{t_1}{x_1}, \dots, \frac{t_r}{x_r} \right) dx_1 \dots dx_r \tag{3.4}$$

Where ρ_i and $\sigma_i (i = 1, 2, \dots, r)$ are non zero real numbers of the same sign and all the integrals involved in equations (3.1) to (3.4) are assumed to be absolutely convergent. Also in (3.4) $k \left(\frac{1}{sx^\sigma} \right)$ stands for $k \left(s_1 x_1^{\rho_1}, \dots, s_r x_r^{\rho_r} \right)$ and so on.

Proof: Applying the formula (2.5) to the pair of equations (3.2) and (3.4), we get

$$\int_0^\infty \dots \int_0^\infty f(x^\sigma) \theta(s_1, \dots, s_r, x_1, \dots, x_r, \rho_1, \dots, \rho_r) dx_1 \dots dx_r = \int_0^\infty \dots \int_0^\infty \prod_{i=1}^r \left(x_i^{\frac{1}{\rho_i}-1} \right) g \left(x^{\frac{1}{\rho}} \right) k \left(sx^{\frac{1}{\rho}} \right) \psi(x) dx_1 \dots dx_r \tag{3.5}$$

Now changing the variables of integrations on the right hand side of (3.5) slightly and interpreting the result thus obtained with the help of (3.1), we easily arrive at (3.3) after a little simplification.

If in the above theorem, we replace ρ_i and $h_i (i = 1, 2, \dots, r)$ and take

$$g(x) = \prod_{i=1}^r \left\{ x_i^{h_i(c_i+1)-1} \right\},$$

$$\phi(x) = \prod_{i=1}^r \left\{ \frac{1}{\Gamma(1-\alpha_i)} {}_2F_1(\alpha_i, \beta_i + m; \beta_i; x_i) \right\}$$

And T to be multidimensional A -function transform defined by (1.13), the right hand side of equation (3.4) assumes the following form:

$$\prod_{i=1}^r \left\{ \frac{t_i^{\gamma_i}}{\Gamma(1-\alpha_i)} \right\} \int_{t_1}^\infty \dots \int_{t_r}^\infty \prod_{i=1}^r \left\{ x_i^{c_i-\gamma_i-1} {}_2F_1 \left(\alpha_i, \beta_i + m; \beta; \frac{t_i}{x_i} \right) \right\} I \left(s_1 x_1^{h_1}, \dots, s_r x_r^{h_r} \right) dx_1 \dots dx_r \tag{3.6}$$

On evaluation the above integral with the help of known results ([4],p.398,eq.(2)) and ([3],p.105,eq.(1)) and the definition of multivariable A -function (1.1) and substituting the values of $\theta (s_1, \dots, s_r, x_1, \dots, x_r, \rho_1, \dots, \rho_r)$ thus obtained, in the right hand side of equation (3.3), we obtain the following interesting corollary with the help of equations (3.1) and (3.3).

Corollary1. If $h_i > 0, \text{Re}(1 - \alpha_i) > m, m \in W$,(the set of whole number) $\sigma_i (i = 1, 2, \dots, r)$ are non-zero real numbers of the same sign.

$$\beta_i \neq 0, -1, -2, \dots; \text{Re}(C_i) \geq 0$$

$$f(x) = \begin{cases} o(x_i^{A_i}), \text{ for small values of } x_i \\ o(x_i^{B_i} e^{-C_i x_i}) \text{ for large values of } x_i \end{cases}; i = 1, 2, \dots, r$$

The multivariable A -function satisfy the conditions corresponding appropriately to those given by Srivastava and Panda ([18], p.130, eqs. (1.1) and (1.15)), then

$$I \left\{ \prod_{i=1}^r x_i^{h_i(c_i+1)-1} \psi(x^h); \begin{matrix} 0, n_2, \dots, 0, n_r; (m', n') \dots; (m^{(r)}, n^{(r)}), \\ p_2, q_2, \dots; p_r, q_r; (p', q') \dots; (p^{(r)}, q^{(r)}) \end{matrix} \right.$$

$$\left. \begin{matrix} (\alpha_{2j}, \alpha_{2j}, \alpha_{2j})_{1, p_2} \dots; (\alpha_{rj}, \alpha_{rj}, \alpha_{rj})_{1, p_r}; (\alpha_j, \alpha_j)_{1, p-1} \dots; (\alpha_j^{(r)}, \alpha_j^{(r)})_{1, p^{(r)-1}} \\ (b_{2j}, \beta_{2j}, \beta_{2j})_{1, q_2} \dots; (b_{rj}, \beta_{rj}, \beta_{rj})_{1, q_r}; (b_j, \beta_j)_{1, q-1} \dots; (b_j^{(r)}, \beta_j^{(r)})_{1, q^{(r)-1}} \end{matrix} \right\} =$$

$$\left\{ \begin{matrix} \left(1 - \alpha_1 + \gamma_1 - c_1, \frac{1}{h_1} \right) \dots; \left(1 - \alpha_r + \gamma_r - c_r, \frac{1}{h_r} \right) \\ \left(1 - \beta_1 + \gamma_1 - m - c_1, \frac{1}{h_1} \right) \dots; \left(1 - \beta_r + \gamma_r - m - c_r, \frac{1}{h_r} \right) \end{matrix}; S_1, \dots, S_r \right\} =$$

$$\frac{1}{\prod_{i=1}^r (\beta_i)_m} I \left\{ \prod_{i=1}^r x_i^{h_i(c_i+1)-1} f(x^{h\sigma}); \begin{matrix} 0, n_2, \dots, 0, n_r; (m', n') \dots; (m^{(r)}, n^{(r)}), \\ p_2, q_2, \dots; p_r, q_r; (p', q') \dots; (p^{(r)}, q^{(r)}) \end{matrix} \right.$$

$$\left. \begin{matrix} (\alpha_{2j}, \alpha_{2j}, \alpha_{2j})_{1, p_2} \dots; (\alpha_{rj}, \alpha_{rj}, \alpha_{rj})_{1, p_r}; (\alpha_j, \alpha_j)_{1, p-1} \dots; (\alpha_j^{(r)}, \alpha_j^{(r)})_{1, p^{(r)-1}} \\ (b_{2j}, \beta_{2j}, \beta_{2j})_{1, q_2} \dots; (b_{rj}, \beta_{rj}, \beta_{rj})_{1, q_r}; (b_j, \beta_j)_{1, q-1} \dots; (b_j^{(r)}, \beta_j^{(r)})_{1, q^{(r)-1}} \end{matrix} \right\}$$

$$\left\{ \begin{matrix} \left(\gamma_1 - c_1, \frac{1}{h_1} \right) \dots; \left(\gamma_r - c_r, \frac{1}{h_r} \right) \\ \left(1 - \beta_1 + \gamma_1 - c_1, \frac{1}{h_1} \right) \dots; \left(1 - \beta_r + \gamma_r - c_r, \frac{1}{h_r} \right) \end{matrix}; S_1, \dots, S_r \right\} \quad (3.7)$$

Where

$$\psi(t_1, \dots, t_r) = I \left\{ f(x); t_1, \dots, t_r \right\} = \prod_{i=1}^r \left\{ \frac{t_i^{\gamma_i}}{\Gamma(1 - \alpha_i)} \right\}$$

$$\int_0^{t_1} \dots \int_0^{t_r} \prod_{i=1}^r \left\{ x_i^{\gamma_i} {}_2F_1 \left(\alpha_i, \beta_i + m; \beta_i; \frac{x_i}{t_i} \right) \right\} f(x^\sigma) dx_1 \dots dx_r \quad (3.8)$$

Provided that

$$\text{Re} \left(c_i - \gamma_i + \frac{1}{h_i} \frac{b_j^{(i)} - 1}{B_j^{(i)}} \right) < 0; 1 \leq j \leq n_i$$

$$\text{Re} \left(\sigma_i B_i + c_i + 1 + \frac{1}{h_i} \frac{b_j^{(i)} - 1}{B_j^{(i)}} \right) < 0; 1 \leq j \leq n_i$$

$$\text{Re} \left(\sigma_i A_i + c_i + 1 + \frac{1}{h_i} \frac{d_k^{(i)} - 1}{D_k^{(i)}} \right) < 0; 1 \leq k \leq m_i$$

$$\operatorname{Re}(\sigma_i A_i + \gamma_i + 1) > 0; i = 1, 2, \dots, r$$

Theorem2. If

$$\tau(s_1, \dots, s_r) = T \left\{ \psi(u^\rho) g(u); s_1, \dots, s_r \right\} = \int_0^\infty \dots \int_0^\infty k(su) \psi(u^\rho) g(u) du_1 \dots du_r \tag{3.9}$$

And

$$\psi(t_1, \dots, t_r) = N \left\{ f(x^\sigma); t_1, \dots, t_r \right\} = \prod_{i=1}^r \left(t_i^{\gamma_i} \int_{t_i}^\infty \dots \int_{t_r}^\infty f(x^\sigma) \prod_{i=1}^r \left(x_i^{-\gamma_i-1} \right) \phi \left(\frac{t_1}{x_1}, \dots, \frac{t_r}{x_r} \right) \right\} dx_1 \dots dx_r \tag{3.10}$$

then

$$\tau(s_1, \dots, s_r) = \frac{1}{\rho_1 \dots \rho_r} \int_0^\infty \dots \int_0^\infty f(x^\sigma) \theta(s_1, \dots, s_r, t_1, \dots, t_r, \rho_1, \dots, \rho_r) dx_1 \dots dx_r \tag{3.11}$$

Where

$$\theta(s_1, \dots, s_r, t_1, \dots, t_r, \rho_1, \dots, \rho_r) = Y \left\{ \prod_{i=1}^r \left(x_i^{\rho_i} \right) g \left(x^\rho \right) k \left(sx^\rho \right); t_1, \dots, t_r \right\} = \prod_{i=1}^r \left(t_i^{-\gamma_i-1} \right) \int_0^{t_1} \dots \int_0^{t_r} \prod_{i=1}^r \left(x_i^{\rho_i} \right) g \left(x^\rho \right) k \left(sx^\rho \right) \phi \left(\frac{x_1}{t_1}, \dots, \frac{x_r}{t_r} \right) dx_1 \dots dx_r \tag{3.12}$$

Where ρ_i and $\sigma_i (i = 1, 2, \dots, r)$ are non zero real numbers of the same sign and all the integrals involved in equations (3.1) to (3.4) are assumed to be absolutely convergent.

Proof: The proof of the above theorem can be easily developed on the lines similar to those of theorem 1.

Again, if in theorem 2, we replace ρ_i by $h_i (i = 1, 2, \dots, r)$ and take

$$g(x) = \prod_{i=1}^r \left\{ x_i^{h_i(c_i+1)-1} \right\}$$

$$\phi(x) = \prod_{i=1}^r \left\{ \frac{1}{\Gamma(1-\alpha_i)} {}_2F_1(\alpha_i, \beta_i + m; \beta_i; x_i) \right\}$$

And T to be multidimensional A -function transform defined by (1.13), then proceeding on the lines similar to those of corollary 1, we obtain the following interesting corollary:

Corollary2. . If $h_i > 0, \operatorname{Re}(1-\alpha_i) > m, m \in W$,(the set of whole number) $\sigma_i (i = 1, 2, \dots, r)$ are non-zero real numbers of the same sign.

$$\beta_i \neq 0, -1, -2, \dots; \operatorname{Re}(c_i) \geq 0$$

$$f(x) = \begin{cases} o(x_i^{A_i}), \text{ for small values of } x_i \\ o(x_i^{B_i} e^{-C_i x_i}) \text{ for large values of } x_i \end{cases}; i = 1, 2, \dots, r$$

The multivariable A -function satisfy the conditions corresponding appropriately to those given by Srivastava and Panda ([18],p.130, eqs. (1.1) and (1.15)), then

$$I \left\{ \prod_{i=1}^r x_i^{h_i(c_i+1)-1} \psi(x^h); \begin{matrix} 0, n_2, \dots, 0, n_r; (m', n') \dots; (m^{(r)}, n^{(r)}), \\ p_2, q_2, \dots; p_r, q_r; (p', q') \dots; (p^{(r)}, q^{(r)}), \end{matrix} \right.$$

$$\left. \begin{matrix} (\alpha_{2j}, \alpha_{2j}, \alpha_{2j})_{1, p_2} \dots; (\alpha_{rj}, \alpha_{rj}, \alpha_{rj})_{1, p_r}; (\alpha_j, \alpha_j)_{1, p-1} \dots; (\alpha_j^{(r)}, \alpha_j^{(r)})_{1, p^{(r)-1}} \\ (b_{2j}, \beta_{2j}, \beta_{2j})_{1, q_2} \dots; (b_{rj}, \beta_{rj}, \beta_{rj})_{1, q_r}; (b_j, \beta_j)_{1, q-1} \dots; (b_j^{(r)}, \beta_j^{(r)})_{1, q^{(r)-1}} \end{matrix} \right.$$

$$\left. \left\{ \begin{matrix} \left(-\gamma_1 - c_1, \frac{1}{h_1}\right) \dots \left(-\gamma_r - c_r, \frac{1}{h_r}\right) \\ \left(\beta_1 - \gamma_1 - c_1 - 1, \frac{1}{h_1}\right) \dots \left(\beta_r - \gamma_r - c_r - 1, \frac{1}{h_r}\right) \end{matrix} ; s_1, \dots, s_r \right\} =$$

$$\frac{1}{r} \prod_{i=1}^m (\beta_i) \left\{ \prod_{i=1}^r x_i^{h_i(c_i+1)-1} f(x^{h\sigma}); \begin{matrix} 0, n_2, \dots, 0, n_r; (m', n') \dots; (m^{(r)}, n^{(r)}), \\ p_2, q_2, \dots; p_r, q_r; (p', q') \dots; (p^{(r)}, q^{(r)}), \end{matrix} \right.$$

$$\left. \begin{matrix} (\alpha_{2j}, \alpha_{2j}, \alpha_{2j})_{1, p_2} \dots; (\alpha_{rj}, \alpha_{rj}, \alpha_{rj})_{1, p_r}; (\alpha_j, \alpha_j)_{1, p-1} \dots; (\alpha_j^{(r)}, \alpha_j^{(r)})_{1, p^{(r)-1}} \\ (b_{2j}, \beta_{2j}, \beta_{2j})_{1, q_2} \dots; (b_{rj}, \beta_{rj}, \beta_{rj})_{1, q_r}; (b_j, \beta_j)_{1, q-1} \dots; (b_j^{(r)}, \beta_j^{(r)})_{1, q^{(r)-1}} \end{matrix} \right.$$

$$\left. \left\{ \begin{matrix} \left(\alpha_1 - \gamma_1 - c_1 - 1, \frac{1}{h_1}\right) \dots \left(\alpha_r - \gamma_r - c_r - 1, \frac{1}{h_r}\right) \\ \left(\beta_1 - \gamma_1 - c_1 + m - 1, \frac{1}{h_1}\right) \dots \left(\beta_r - \gamma_r - c_r + m - 1, \frac{1}{h_r}\right) \end{matrix} ; s_1, \dots, s_r \right\} \tag{3.13}$$

Where

$$\psi(t_1, \dots, t_r) = N \left\{ f(x); t_1, \dots, t_r \right\} = \prod_{i=1}^r \left\{ \frac{t_i^{\gamma_i}}{\Gamma(1 - \alpha_i)} \right\}$$

$$\int_{t_1}^{\infty} \dots \int_{t_r}^{\infty} \prod_{i=1}^r \left\{ x_i^{-\gamma_i-1} {}_2F_1 \left(\alpha_i, \beta_i + m; \beta_i; \frac{t_i}{x_i} \right) \right\} f(x^\sigma) dx_1 \dots dx_r \tag{3.14}$$

Provided that

$$\operatorname{Re} \left(c_i + \gamma_i + \frac{1}{h_i} \frac{d_j^{(i)}}{D_j^{(i)}} \right) > 0; 1 \leq j \leq m_i$$

$$\operatorname{Re} \left(\sigma_i B_i + c_i + 1 + \frac{1}{h_i} \frac{b_j^{(i)} - 1}{B_j^{(i)}} \right) < 0; 1 \leq j \leq n_i; i = 1, 2, \dots, r$$

$$\operatorname{Re} \left(\sigma_i A_i + c_i + 1 + \frac{1}{h_i} \frac{d_k^{(i)}}{D_k^{(i)}} \right) > 0; 1 \leq k \leq m_i$$

$$\operatorname{Re} (\sigma_i A_i + \beta_i - \gamma_i + m) > 0; i = 1, 2, \dots, r$$

The one and two-dimensional analogous of Theorems 1 and 2 can easily be deduced but since the theorems contain a reasonably detailed analysis of the multidimensional case we prefer to omit their details. The corollaries 1 and 2 given earlier are also new. They give rise to interesting theorems involving multidimensional analogues of fractional integral operators defined by Kober [8], Riemann-Liouville [4] and Weyl [4] and simpler multidimensional integral transforms, on suitably specializing the fractional integral operators and multidimensional A -function transform involved therein. Again,

the one and two-dimensional analogues of corollaries 1 and 2, yield theorems essentially similar to those given earlier by Gupta, Goyal and Handa ([6], p.165-170), Handa ([7], p. 200-204), Kalla ([9], p. 1008-1010, [10] p. 54-56) and Mathur ([12] p.108-121).

REFERENCES

- [1] Erdelyi, A.; on fractional integration and its application to the theory of Hankel transforms, *Quart. J. Math. Oxford Ser. (2)2*, (1920), 293-303.
- [2] Erdelyi, A.; on some fractional transformations, *Univ. Politec. Torino. Rend. Sem. Mat.* 10, (1950), 217-234.
- [3] Erdelyi, A. et.al.; *Higher Transcendental Functions*, vol. I, McGraw-Hill, New York (1953).
- [4] Erdelyi, A. et.al.; *Tables of Integral Transforms*, vol. II, McGraw-Hill, New York (1954).
- [5] Gautam, G.P. and Goyal, A.N.; On Fourier kernels and asymptotic expansion. *Indian J. Pure Appl. Math.* 12,, (1981), 1109-1105.
- [6] Gupta, K.C. , Goyal, S.P. and Handa, S.; Fractional integral operators involving two variables, *Univ. Nac. Tucuman Rev. Ser. A* 26, (1976), 159-171.
- [7] Handa, S.; A study of generalized functions of one and two variables, *Indian J. of Math.*, (1976), 11-16.
- [8] Kober, H.; on fractional integrals and derivatives, *Quart. J.Math. Oxford Ser. 2(II)*, (1940), 193-211.
- [9] Kalla, S.L.; Some theorems on fractional integration, *Proc. Nat. Acad. Sci. India Sect. a* 36, (1966), 1007-1012.
- [10] Kalla, S.L.; Some theorems on fractional integration II, *Proc. Nat. Acad. Sci. India Sect.* 39, (1969), 44-56.
- [11] Kalla, S.L.; Fractional integration operators involving generalized hypergeometric functions, *Univ. Nac. Tucuman. Rev. Ser. A* 20, (1970), 93-100.
- [12] Mathur, S.L.; Astudy of boundary value problems and special functions, *Proc. Acad. Sci India Sect.A* 43, (1974), 56-61.
- [13] Saxena, R.K.; on fractional integral operators, *Math. Zeitscher.* 95, (1967), 288-291.
- [14] Saxena, R.K. and Modi, G.C.; Multidimensional fractional integration operators associated with hypergeometric kernels, *Nat. Acad. Sci. Lett.* 13, (1980), 153-158.
- [15] Sneddon, I.N.; *Mixed Boundary Value Problems in Potential Theory*, North-Holland Publishing Co., Amsterdam (1966).
- [16] Srivastava, H.M. and Buschmann R.G.; Composition of fractional integral operators involving Fox's H -function, *Acta Mexicana Ci Tecn* 7, (1973), 21-28.
- [17] Srivastava, H.M. and Panda, R.; Some bilateral generating functions for a class of generalized polynomials, *J. Reine Angew Math.* 283/284, (1976)265-274.
- [18] Srivastava, H.M. and Panda, R.; Expansion theorems for the H -function of several complex variables, *J.Reine Angew Math* 288, (1976), 129-145.
- [19] Srivastava, H.M. and Panda, R.; Certain multidimensional integral transformations I and II, *Nederl. Akad Wetensch Proc. Ser. A* 81, (1978), 118-141.
- [20] Srivastava, H.M., Gupta, K.C. and Goyal, S.P.; the H -function of One and Two Variables With Applications, South Asian Publishers Pvt. Ltd., New Delhi and Madras (1982).