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Variational Iteration Sumudu Transform Method for Solving Fractional Nonlinear Gas Dynamics Equation

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Abstract: *In this paper, generalized nonlinear fractional gas dynamics equation has been solved by using fractional variational iteration Sumudu transform method (VISTM). This technique was extended to derive analytical solutions in the form of a series for this equation which converges to the exact solution of the problem. It is indicated that the solutions obtained by the (VISTM) are reliable, very efficient, and professional techniques for solving different kinds of nonlinear fractional differential equation.*

Keywords: *Fractional calculus, Sumudu transform, Fractional variational iteration method, Variational iteration Sumudu transform method (VISTM), Nonlinear fractional gas dynamics equation.*

1. Introduction

Nonlinear partial differential equations are useful in describing the various Phenomena in disciplines. The variational iteration method was proposed by He [1] and was successfully applied to autonomous ordinary differential equation [2] to nonlinear partial equations with variable coefficients [3] to Schrodinger-KdV, generalized KdV and shallow water equations [4] to linear Helmholtz partial differential equation [5] recently to nonlinear fractional differential equations with Caputo differential derivative [6, 7] and to other fields, [8]. The variational iteration method gives rapidly convergent successive approximations of the exact solution if such a solution exists; otherwise a few approximations can be used for numerical purposes. The method is effectively used in [2-4, 9-11] and the references there in Jafari et al [9, 10] applied the variational iteration method to the gas dynamics equation and Stefan problem. The objective of this paper is to extend the application of the fractional Variational iteration method combined with Sumudu transform method (VISTM) to solve nonlinear fractional gas dynamics equation.

2. Fractional Calculus

The fractional calculus has found diverse applications in various scientific and technological fields [12, 13], such as thermal engineering, acoustics, electromagnetism, control, robotics, viscoelasticity, diffusion, edge detection, turbulence, signal processing, and many other physical processes. Fractional differential equations (FDEs) have also been applied in modeling many physical, engineering problems, and fractional differential equations in nonlinear dynamics [14, 15]. In this section, we mention the following basic definitions of fractional calculus which are used further in the present paper:

Definition 1: The Riemann-Liouville fractional integral operator of order $\alpha \succ 0$, of a function $f(t) \in c_p$ and $p \geq -1$ is defined as [16].

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, (\alpha > 0),\ I^0 f(t) = f(t).$$

For the Riemann- Louville fractional integral we have:

$$I^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$$

Definition 2: The Sumudu transform of the Caputo fractional derivative is defined as follows [17]:

$$S[D_t^{\alpha} f(t)] = u^{-\alpha} S[f(t)] - \sum_{k=0}^{m-1} u^{-\alpha+k} f^{(k)}(0+), (m-1 < \alpha \le m)$$

Definition 3: The Mittage-Leffler function $E_{\alpha}(z)$ with $\alpha > 0$ is defined by the following series representation, valid in the whole complex plane [18]:

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}$$

Definition 4: Fractional derivative of compounded functions [19, 20] is defined as

$$d^{\alpha} f \cong \Gamma(\alpha+1) df, 0 < \alpha < 1$$

Definition 5: The integral with respect to $(dx)^{\alpha}$ is defined as the solution of the fractional differential equation [19, 20]:

$$dy \cong f(x)(dx)^{\alpha}, x \ge 0, y(0) = 0, 0 < \alpha < 1$$

Definition 6: In early 90's, Watugala [21] introduced a new integral transform, named the Sumudu transform and applied it to the solution of ordinary differential equation in control engineering problems. The Sumudu transform is defined over the set of functions:

$$A = \left\{ f(t) : \exists M, \tau_1, \tau_2 > 0, \left| \; f(t) \; \right| < M e^{|t|/\tau_j}, \; if \; t \in (-1)^j \times [0, \infty) \right\}$$

by the following formula

$$G(u) = S[f(t); u] = \int_0^{\infty} f(ut) e^{-t} dt, u \in (-\tau_{1,}, \tau_2).$$

Some of the properties were established by Weerakoon in [22, 23]. In [24], by Asiru, further fundamental properties of this transform were also established. Similarly, this transform was applied to the one-dimensional neutron transport equation in [25] by Kadem. In fact it was shown that there is a strong relationship between Sumudu and other integral transforms; see Kılıcman et al. [26]. In particular the relation between Sumudu transform and Laplace transforms was proved in Kılıcman and Gadain [27]. Further, the Sumudu transform was extended to the distributions and some of their properties were also studied in Kılıcman and Eltayeb [28]. Recently, this transform is applied to solve the system of differential equations; see Kılıcman et al. in [29].

3. FRACTIONAL VARIATIONAL ITERATION METHOD (FVIM)

According to the Vim's rules [30 -32, 33] one need to follow three steps to determine the variational iteration formula:

a. Establishing the correction functional,
b. Identifying the Lagrange multipliers,
c. Determining the initial iteration.

To describe the solution procedure of the fractional variational iteration method, we consider the following fractional differential equation:

$$ {}_0^c D_t^{\alpha} U + R[U(t)] + N[U(t)] = K(t), 0 < \alpha \tag{1}$$

Where R is a linear operator, N is a nonlinear operator and $K(t)$ is a given continuous function. Construct a correction functional:

$$U_{n+1} = U_n + \int_0^t \lambda(x,\tau) [{}_0^c D_{\tau}^{\alpha} U + R[U(\tau)] + N[U(\tau)] - K(\tau)] d\tau \tag{2}$$

Where $\lambda(x,\tau) = \frac{(-1)^{\alpha}}{\Gamma(\alpha)} (\tau - t)^{\alpha-1}$ multiplier or a weighted function for any fractional order $\alpha > 0$,

$$U_{n+1} = U_n + \int_0^t \frac{(-1)^{\alpha}}{\Gamma(\alpha)} (\tau - t)^{\alpha-1} [{}_0^c D_{\tau}^{\alpha} U + R[U(\tau)] + N[U(\tau)] - K(\tau)] d\tau \tag{3}$$

$$U_{n+1} = U_n - {}_0 I_t^{\alpha} [{}_0^c D_t^{\alpha} U + R[U(t)] - K(t)] \tag{4}$$

The above formula is also valid for the FDEs with the R-L derivative.

It is obvious that the sequential approximations $U_k, k \geq 0$ can be established by determining λ, a general Lagrange's multiplier which can be identified optimally with the variational theory. The function $\tilde{U}_n$ is a restricted variation which means $\partial \tilde{U}_n = 0$. Therefore, we first designate the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximations $U_{n+1}(x,t), n \geq 0$ of the solution $U(x,t)$, will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function U_0. The initial values are usually used for choosing the zeroth approximation U_0. With λ determined, then several approximations, $U_k, k \geq 0$ follows immediately [34]. Consequently the exact solution may be procured by using:

$$U(x,t) = \lim_{n\to\infty} U_n(x,t) \tag{5}$$

4. Fractional Variational Iteration Sumudu Transform Method

In the case of an algebraic equation $f(x) = 0$, the Lagrange multipliers can be evaluated by an iteration formula for finding the solution of the algebraic Equation $f(x) = 0$ that can be constructed as

$$x_{n+1} = x_n + \lambda f(x_n). \tag{6}$$

The optimality condition for the extreme $\frac{\partial x_{n+1}}{\partial x_n} = 0$ Leads to

$$\lambda = -\frac{1}{f'(x_n)}, \tag{7}$$

Where ∂ is the classical variational operator. From (6) and (7), for a given initial value X_0, we can find the approximate solution X_{n+1} by the iterative scheme for (6) as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, f(x_0) \neq 0, n = 0,1,2,..... \tag{8}$$

This algorithm is well known as the Newton-Raphson method and has quadratic convergence. Now, we extend this idea to finding the unknown Lagrange multiplier. The main step is to first take the Sumudu transform to Eq. (1). Then the linear part is transformed into an algebraic equation as follows:

$$\frac{G(u)}{u^{\alpha}} - \frac{U(0)}{u^{\alpha}} .... - \frac{U^{(n-1)}(0)}{u^{\alpha-(n-1)}} + S[R[U(t)] + N[U(t)] - K(t)] = 0, \tag{9}$$

Where

$$G(u) = S[U(t)] = \int_0^{\infty} U(ut)e^{-t}dt, -\tau_1 \prec u \prec \tau_2.$$

The iteration formula of (8) can be used to suggest the main iterative scheme involving the Lagrange multiplier as:

$$G_{n+1} = G_n + \lambda\left(\frac{G(u)}{u^{\alpha}} - \frac{U(0)}{u^{\alpha}} .... - \frac{U^{(n-1)}(0)}{u^{\alpha-(n-1)}} + S[R[U(t)] + N[U(t)] - K(t)]\right). \tag{10}$$

Considering $S[R[U(t)] + N[U(t)]$ as restricted terms, one can derive a Lagrange multiplier as:

$$\partial G_{n+1} = \partial G_n + \partial\left(\lambda \frac{G_n(u)}{u^{\alpha}}\right)$$

$$\partial G_{n+1} = \partial G_n + \frac{1}{u^{\alpha}}(\lambda' G_n + \lambda \partial G_n)$$

This yields the stationary conditions of Eq. (10) as follows;

$$\partial G_n(1+\frac{\lambda}{u^\alpha})=0,$$

$$\frac{1}{u^\alpha}(\lambda' G_n)=0$$

$$\therefore \lambda=-u^\alpha$$

With Eq. (10) and the inverse-Sumudu transform S^{-1}, the iteration formula (9) can be explicitly given as:

$$U_{n+1}=U_n-S^{-1}[u^\alpha(\frac{G(u)}{u^\alpha}-\frac{U(0)}{u^\alpha}..........-\frac{U^{(n-1)}(0)}{u^{\alpha-(n-1)}}+S[R[U(t)]+N[U(t)]-K(t)]\!)] \tag{11}$$

$$U_{n+1}=S^{-1}[u^\alpha(\frac{U(0)}{u^\alpha}..........-\frac{U^{(n-1)}(0)}{u^{\alpha-(n-1)}})-u^\alpha(S[R[U(t)]+N[U(t)]-K(t))], \tag{12}$$

Consequently the exact solution may be procured by using

$$U(x,t)=\lim_{n\to\infty}U_n(x,t) \tag{13}$$

5. An Application

In this paper, we consider the following nonlinear time- fractional gas dynamics equation of the form:

$$D_t^\alpha U+\frac{1}{2}(U^2)_x-U(1-U)=0, t>0, 0<\alpha\le 1, \tag{14}$$

with the initial condition:

$$U(x,0)=e^{-x} \tag{15}$$

where α is parameter describing the order of the fractional derivative. The function $U(x,t)$ is the probability density function, t is the time and x is the spatial coordinate. The derivative is understood in the Caputo sense. The general response expression contains parameter describing the order of the fractional derivative that can be varied to obtain various responses. In the case of $\alpha=1$ the fractional gas dynamics equation reduces to the classical gas dynamics equation. The gas dynamics equations are based on the physical laws of conservation, namely, the laws of conservation of mass, conservation of momentum, conservation of energy etc. The nonlinear fractional gas dynamics has been studied previously by Das and Kumar [35].

Further, we apply the (VISTM) to solve the nonlinear time- fractional gas dynamics equation. The objective of the present paper is to extend the application of the (VISTM) to obtain analytic and approximate solutions to the time-fractional gas dynamics equation.

The Solution

After taking the Sumudu transform on both sides of Eq. (14) the iteration formula of Eq. (14) can be constructed as:

$$G_{n+1}=G_n+\lambda(u)[\frac{G_n}{u^\alpha}-\frac{\tilde{U}_0}{u^\alpha}+S[\frac{1}{2}(\tilde{U}_n^2)_x-\tilde{U}_n(1-\tilde{U}_n)]] \tag{16}$$

where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory, $\tilde{U}_0=0$ and $S[\frac{1}{2}(\tilde{U}_n^2)_x-\tilde{U}_n(1-\tilde{U}_n)]$ is a restricted variation, that is , $\partial\tilde{U}_n=0$

i.e.

$$\partial G_{n+1}=\partial G_n+\partial G_n\frac{\lambda}{u^\alpha} \tag{17}$$

This yields the stationary conditions, which gives $\lambda=-u^\alpha$.
Substituting this value of Lagrangian multiplier in (16) we get the following iteration formula:

$$G_{n+1}=G_n-u^\alpha[\frac{G_n}{u^\alpha}-\frac{U_0}{u^\alpha}+S[\frac{1}{2}(U_n^2)_x-U_n(1-U_n)]]. \tag{18}$$

Applying inverse Sumudu transform on both sides of Eq. (18) we get:

$$U_{n+1} = U_n - S^{-1}(u^{\alpha}[\frac{G_n}{u^{\alpha}} - \frac{U_0}{u^{\alpha}} + S[\frac{1}{2}(U_n^2)_x - U_n(1-U_n)]]) \quad (19)$$

with the initial condition

$U(x,0) = e^{-x}$

i.e.

$U_0 = e^{-x}$,

$$U_1 = e^{-x} + e^{-x}\frac{t^{\alpha}}{\Gamma(\alpha+1)},$$

$$U_2 = e^{-x} + e^{-x}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + e^{-x}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$U_3 = e^{-x} + e^{-x}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + e^{-x}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + e^{-x}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},\ldots$$

Finally, we approximate the analytical solution $U(x,t)$ by:

$$U(x,t) = e^{-x}\sum_{k=0}^{\infty}\frac{(t^{\alpha})^k}{\Gamma(\alpha k+1)} \quad (20)$$

Where $\sum_{k=0}^{\infty}\frac{(t^{\alpha})^k}{\Gamma(\alpha k+1)} = E_{\alpha}(t^{\alpha})$ is the famous Mittage–Leffler function

i.e. Eq. (20) takes the form:

$$U(x,t) = e^{-x}E_{\alpha}(t^{\alpha}) \quad (21)$$

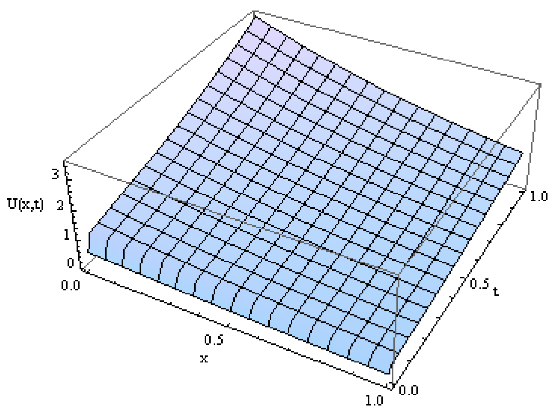
In special case at $\alpha = 1$ we have

$$U(x,t) = e^{-x}\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(k+1)} = e^{-x}e^{t} = e^{t-x}$$

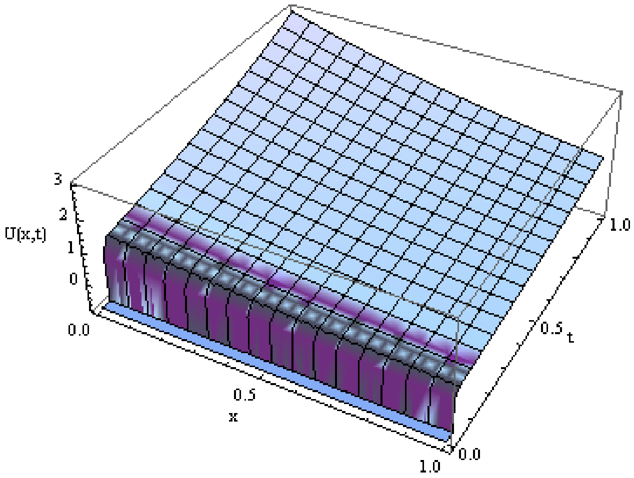
$$\therefore\ U(x,t) = e^{t-x}. \quad (22)$$

Which is the same as given by HPM and HPSTM (see Jafari et al [36] , and Singh et al [37]).

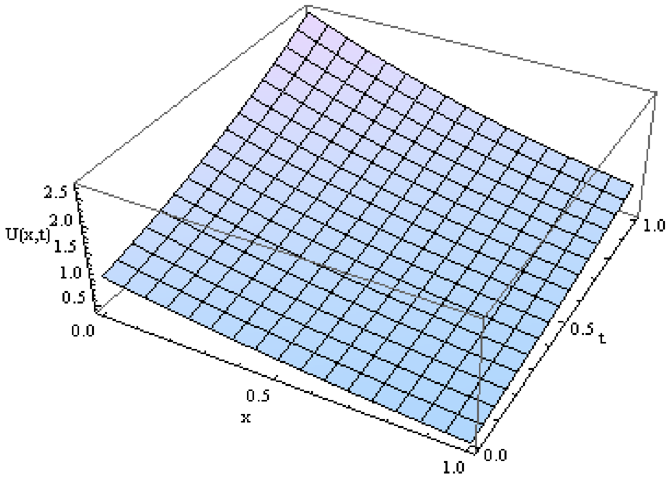
Now, we calculate numerical results of the probability density function $U(x,t)$ for different time-fractional Brownian motions α =1/3,1/2,1 and for various values of t and x .The numerical results for the approximate solution (21) obtained by using VIST M and the exact solution (22) for various values of t ,x ,and α are shown in Figure (a)-(d) . It is observed from Figures (a) and (b) that $U(x,t)$ increase with the increase in t and $U(x,t)$ decrease with the increase in α . Figure 1(c) and figure 2 (d) clearly show that, when α =1, the approximate solution (21) obtained by the present method is very near to the exact solution .It is evident that the efficiency of the present method can be dramatically.



a)

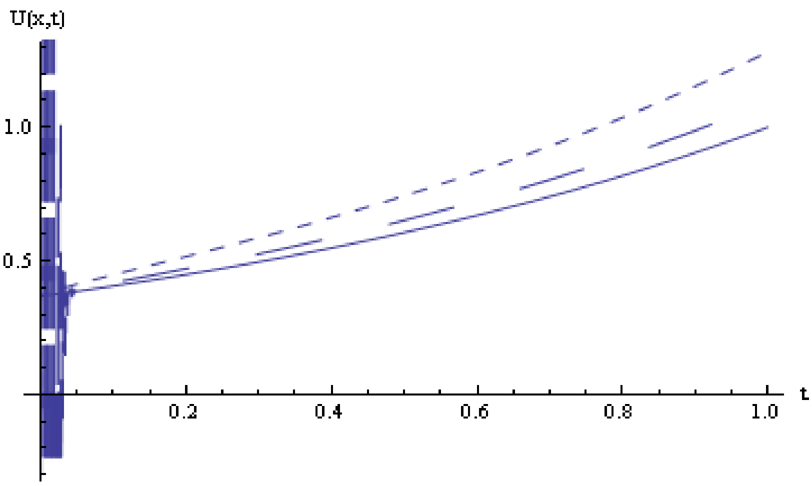


b)



c)

Figure 1. *The behavior of the U (x, t), x and t are obtained when (a) α=0.75, (b) α =0.90 , (c) α=1.*



d)

Figure 1. The behavior of the $U(x, t)$ and t are obtained at (---)α =0.90, (---),α =0.75(----),α =1.

6. Conclusion

In this paper, variational iteration Sumudu transform method (VISTM) is successfully applied for solving nonlinear time- fractional gas dynamics equation. This method is very powerful and efficient techniques for solving different kinds of linear and nonlinear fractional differential equations arising in different fields of science and engineering. With the approach given in this paper, we can easily derive Lagrange multipliers without tedious calculation and new variational iteration formulae can be derived. Some FDEs with the Caputo derivatives are illustrated. The results show the modified method's efficiency compared with other versions of the VIM in fractional calculus. In conclusion, the VISTM may be considered as a nice refinement in existing numerical techniques and might find the wide applications.

REFERENCES

[1] J. He, "A new approach to nonlinear partial differential equations," Communications in Nonlinear Science and Numerical Simulation, vol. 2, no. 4, pp. 230–235, 1997.

[2] J.-H. He, "Variational iteration method for autonomous ordinary differential systems," Applied Mathematics and Computation, vol. 114, no. 2-3, pp. 115–123, 2000.

[3] J.H. He, "Variational principles for some nonlinear partial differential equations with variable coefficients," Chaos, Solitons & Fractals, vol. 19, no. 4, pp. 847–851, 2004.

[4] M. A. Abdou and A. A. Soliman, "New applications of variational iteration method," Physica D, vol. 211, no. 1-2, pp. 1–8, 2005.

[5] S. Momani and S. Abuasad, "Application of He's variational iteration method to Helmholtz equation," Chaos, Solitons & Fractals, vol. 27, no. 5, pp. 1119–1123, 2006.

[6] Z. M. Odibat and S. Momani, "Application of variational iteration method to nonlinear differential equations of fractional order," International Journal of Nonlinear Sciences and Numerical Simulation, vol. 7, no. 1, pp. 27–34, 2006.

[7] S. Momani and Z. Odibat, "Numerical approach to differential equations of fractional order," Journal of Computational and Applied Mathematics, vol. 207, no. 1, pp. 96–110, 2007.

[8] J.-H. He, "Some asymptotic methods for strongly nonlinear equations," International Journal of Modern Physics B, vol. 20, no. 10, pp. 1141–1199, 2006.

[9] H. Jafari, H. Hosseinzadeh, and E. Salehpoor, "A new approach to the gas dynamics equation: an application of the variational iteration method," Applied Mathematical Sciences, vol. 2, no. 48, pp. 2397–2400, 2008.

[10] H. Jafari, A. Golbabai, E. Salehpoor, and Kh. Sayehvand, "Application of variational iteration method for Stefan problem," Applied Mathematical Sciences, vol. 2, no. 60, pp. 3001–3004, 2008.

[11] H. Jafari and A. Alipoor, "A new method for calculating General Lagrange's multiplier in the variational iteration method," Numerical Method for Partial Differential Equations, In press, 2010.

[12] I. Podlubny, "Fractional Differential Equations," Mathematics in Science and Engineering, Academic Press, San Diego, Cal if, USA, vol. 198, 1999.

[13] K. B. Oldham and J. Spanier, "The Fractional Calculus," Academic Press, New York, NY, USA, 1974.

[14] H. Jafari and V. Daftardar-Gejji, "Solving a system of nonlinear fractional differential equations using Adomian decomposition," Journal of Computational and Applied Mathematics, vol. 196, no. 2, pp. 644–651, 2006.

[15] J. G. Lu and G. Chen, "A note on the fractional-order Chen system," Chaos, Solitons & Fractals, vol. 27, no. 3, pp. 685–688, 2006.

[16] I. Podlubny, "Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and some of Their Applications," Mathematics in Science and Engineering, Academic Press, San Diego, Calif, USA, vol. 198, 1999.

[17] V. B. L. Chaurasia and J. Singh, "Application of Sumudu transform in Schrödinger equation occurring in quantum mechanics," Applied Mathematical Sciences, vol. 4, no. 57–60, pp. 2843–2850, 2010.

[18] F. Mainardi, "On the initial value problem for the fractional diffusion-wave equation," in Waves and Stability in Continuous Media, S. Rionero and T. Ruggeeri, Eds., World Scientific, Singapore, pp. 246–251, 1994.

[19] Jumarie, "Laplace's transform of fractional via the Mittag-Leffler function and modified order Riemann–Liouville derivative," Applied Mathematics Letters, vol. 22, pp. 1659-1664, 2009.

[20] Jumarie, "Table of some basic fractional calculus formulae derived from a modified Riemann–Liouvillie derivative for non differentiable functions," Applied Mathematics Letters, vol. 22, pp. 378-385, 2009.

[21] G.K. Watugala, "Sumudu transform - a new integral transform to solve differential equations and control engineering problems," Math. Engg. Indust., vol. 6, no. 4, pp. 319-329, 1998.

[22] S. Weerakoon, "Application of Sumudu transform to partial differential equations," International Journal of Mathematical Education in Science and Technology, vol. 25, no. 2, pp. 277–283, 1994.

[23] S. Weerakoon, "Complex inversion formula for Sumudu transform," International Journal of Mathematical Education in Science and Technology, vol. 29, no. 4, pp. 618–621, 1998.

[24] M. A. Asiru, "Further properties of the Sumudu transform and its applications," International Journal of Mathematical Education in Science and Technology, vol. 33, no. 3, pp. 441–449, 2002.

[25] A. Kadem, "Solving the one-dimensional neutron transport equation using Chebyshev polynomials and the Sumudu transform," Analele Universitatii dinOradea, vol. 12,pp. 153-171, 2005.

[26] A. Kılıcman, H. Eltayeb, and K. A. M. Atan, "A note on the comparison between Laplace and Sumudu transforms," Iranian Mathematical Society, vol. 37, no. 1, pp. 131-141, 2011. 347, no. 5, pp. 848–862, 2010.

[27] A. Kılıcman and H. E. Gadain, "On the applications of Laplace and Sumudu transform," Journal of the Franklin Institute, vol. 4, no. 1-4, pp. 109-118, 2010.

[28] H. Eltayeb, A. Kılıcman, and B. Fisher, "A new integral transform and associated distributions," Integral Transforms and Special Functions, vol. 21, no. 5-6, pp. 367–379, 2010.

[29] A. Kılıc¸man and H. Eltayeb, "A note on integral transforms and partial differential equations," Applied Mathematical Sciences, vol. 4, no. 1-4, pp. 109-118, 2010.

[30] A. Kılıc¸man, H. Eltayeb, and R. P. Agarwal, "On Sumudu transform and system of differential equations," Abstract and Applied Analysis, Article ID598702, 11 pages, 2010.

[31] J. H. He, "Approximate analytical solution for seepage flow with fractional derivatives in porous media," Comput. Method. Appl. M. 167 57-68, 1998.

[32] J. H. He, "Variational iteration method - a kind of non-linear analytical technique: Some examples, " International Journal of nonlinear Mechanics 699-708, 1999.

[33] J. H. He, "Variational iteration method - Some recent results and new interpretations," Journal of Computational and Applied Mathematics. 3-17, 2007.

[34] J. H. He, X. H. Wu "Variational iteration method: new development and applications," Comput. Math. Appl. 881-894, 2007.

[35] S. Das, and R. Kumar, "Approximate analytical solutions of fractional gas dynamics equations," Applied Mathematics and Computation, vol. 217, pp. 9905-9915, 2011.

[36] H. Jafari , M. Zabihi and M. Saidy,"Application of Homotopy-perturbation method for Solving gas dynamics equation," Applied Mathematical Sciences, Vol. 2, no. 48, 2393 – 2396,2008.

[37] J Singh, D Kumar, and A. Kılıçman," Homotopy perturbation method for fractional gas dynamics equation using sumudu transform," Hindawi Publishing Corporation, Vol.8, 2013.

[38] A. A. M. Arafa, "Fractional Differential Equations in Description of Bacterial Growth," Differential Equations & Dynamical Syststem, **21**, 205-214 (2013).

[39] A.A. M. Arafa, S.Z. Rida, M. Khalil, "Approximate analytical solutions of Schnakenberg systems by homotopy analysis method," Applied Mathematical Modeling, **37**, 2189-2196 (2013).

[40] A.A.M. Arafa,S.Z. Rida, A.A. Mohammadein,and H.M. Ali, "Solving Nonlinear Fractional Differential Equation by Generalized Mittag-Leffler Function Method," Commun Theor. Phys., **59**, 661-663 (2013).

[41] A. M. Arafa, S.Z. Rida, H. Mohamed, "Approximate analytical solutions of Schnakenberg systems by homotopy analysis method," Applied Mathematical Modelling, **36**, 4789-4796 (2012).

[42] A. A. M. Arafa. "Series Solutions of Time-Fractional Host-Parasitoid Systems," J Stat Phys, **145**, 1357-1367 (2011).

[43] 43 . S. Z. Rida and A. A. M. Arafa, "New Method for Solving Linear Fractional Differential Equations," International journal of differential equations, Article ID 8141328**,2011**.

[44] A. A. M. Arafa, S. Z. Rida, H. Mohamed, "Homotopy Analysis Method for Solving Biological Population Model," Commun. Theor. Phys., **56**, 797-800 (2011).

[45] A. A. M. Arafa, S. Z. Rida, H. Mohamed, "Homotopy Analysis Method for Solving Biological Population Model," Commun. Theor. Phys., **56**, 797-800 (2011).

[46] A.A.M. Arafa , S. Z. Rida , M.Khalil , "Fractional modeling dynamics of HIV and CD4+ T-cells during primary infection," Nonlinear Biomedical Physics, **6**, 1 (2012).

[47] A. A. M. Arafa, S.Z. Rida, and *Hegagi Mohamed*, An Application of the Homotopy Analysis Method to the Transient Behavior of a Biochemical Reaction Model, Inf. Sci. Lett. **3**, No. 1, 29-33 (2014).

[48] *A.A. M. Arafa, I. M. Hanafy and M. I. Gouda***,** Numerical Simulations of Bromsulphthalein Test for Human Liver, Math. Sci. Lett. 3, No. 2, 75-79 (2014).

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